

# CHAIN RULES FOR LINEAR OPENNESS IN METRIC SPACES. APPLICATIONS TO PARAMETRIC VARIATIONAL SYSTEMS

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**Abstract:** In this work we present a general theorem concerning chain rules for linear openness of set-valued mappings acting between metric spaces. As particular cases, we obtain classical and also some new results in this field of research, including the celebrated Lyusternik-Graves Theorem. The applications deal with the study of the well-posedness of the solution mappings associated to parametric variational systems. Sharp estimates for the involved regularity moduli are given.

**Keywords:** composition of set-valued mappings · linear openness · metric regularity · Aubin property · implicit multifunctions · local composition-stability · parametric variational systems

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## 1 Introduction

The openness at linear rate was, over time, an intensively studied issue, having as first landmark one of the most profound results in the theory of linear operators – Banach’s Open Mapping Principle. Published by J. Schauder in 1930 and by S. Banach in his famous book from 1932, it proved itself to be very useful in a wide variety of problems, and it can be formulated in order to emphasize the equivalent properties of linear openness and metric regularity. The wide applicability of this principle resulted in multiple attempts to extend it, creating another two classical results in nonlinear analysis, for strictly differentiable functions: the Tangent Space Theorem, proved by L.A. Lyusternik [13] in 1934, and the Surjection Theorem, proved by L.M. Graves [10] in 1950. Another landmark was the extension of the research to the case of set-valued mappings with closed and convex graph and this was done by Ursescu [17] in 1975 and Robinson [15] in 1976, respectively. The celebrated Robinson-Ursescu Theorem was followed by several works of Robinson and Milyutin in 1970s and 1980s concerning the preservation of regularity (and linear openness) under functional perturbation and the corresponding applications in the study of generalized equations. Without

being exhaustive, the following list contains other major contributors to the further development of the field: J. P. Aubin, A. Dontchev, H. Frankowska, A. Ioffe, A. S. Lewis, B. S. Mordukhovich, J.-P. Penot, R. T. Rockafellar.

However, it was clearly emphasized in the works of Ioffe [11] and Dontchev and Rockafellar [7] that the interrelated properties of metric regularity, Aubin and openness at linear rate have an intimate metric character. Generally speaking, there are two main techniques to obtain metric regularity results. The first one, going back to the original work of Lyusternik, is based on a constructive iterative procedure, while the second one uses Ekeland Variational Principle. Despite the fact that Ekeland Variational Principle works on complete metric spaces, its use in this direction generates results stated on Banach spaces and the arguments of the proofs are given by means of contradiction. Up to 2000s, the main literature on this topic considers both above mentioned methods, but the general framework is that of Banach spaces; after that, the effort to avoid the linear structure and to work in purely metric setting became more and more desirable.

This evolution could be better observed on a concrete fundamental result - the Lyusternik-Graves Theorem - and its successive extensions. The classical Lyusternik-Graves Theorem establishes a metric regularity of a strictly differentiable function from a surjectivity condition of its derivative. Moreover, it was observed that, in fact, it can be deduced from a result of Graves concerning the preservation of metric regularity under perturbations by surjective continuous linear operators. Subsequently, this was extended by Milyutin to the case where the linear perturbation is replaced by a linearly open single-valued map. In 1996, Ursescu [18] was the first to obtain a fully set-valued extension of the above results, keeping the setting of Banach spaces. A crucial observation is suggested by the second proof of Theorem 6, p. 520 from the milestone survey of Ioffe ([11]). More precisely, it became apparent that the Lyusternik iteration process can be successfully used when the income space is a complete metric space and the outcome space has a linear structure with shift-invariant metric. However, the perturbations considered by Ioffe is more restrictive than those in Ursescu's work.

Recently, following an idea of Arutyunov ([3]) concerning an extension of Nadler fixed point theorem, Ioffe [12] and Dontchev and Frankowska [5] gave, on one side, openness results for set-valued compositions, and, on the other side, fully metrical extensions, without any linear structure, of Lyusternik-Graves Theorem.

The purpose of this work is to present a very general theorem concerning chain rules for linear openness of set-valued mappings acting between metric spaces. In this way we enter into a dialog with both papers [12] and [5], following the research of the authors previously developed on Banach spaces ([8]). In some particular cases, we obtain classical and also some new results in this field of research. For instance, the celebrated Lyusternik-Graves Theorem appears as a particular case of composition and our approach brings into light the role of the shift-invariant property of the metric on the outcome space. The same mechanism is shown to be available in some situations on complete Riemannian manifolds, in relation to another particular case of composition.

As application, we study the well-posedness of the solution mappings associated to parametric variational systems, giving sharp estimates for the involved regularity moduli. To this aim we introduce a local chain stability notion which preserve the Aubin property for compositions of multifunctions.

The paper is organized as follows. After a short section of preliminaries, we present the main result of this work which refers to the openness at linear rate of general set-valued compositions. Then we discuss several important particular cases where the assumptions of the main results are fulfilled. Firstly, we take into consideration the case where the composition map is the sum and

this allows us to clearly emphasize the role of the shift-invariance property of the metric for the formulation of the conclusion in Lyusternik-Graves Theorem. We underline the fact that the same type of assertions can be locally obtained without shift-invariance property but with an appropriate change of openness rates. Secondly, we analyze the situation when the composition map is the distance function. We identify two major cases where this map has the desired properties: the settings of normed vector spaces and of complete Riemannian manifolds. As a by-product we provide another proof of a particular case of [5, Theorem 5].

The last section concerns some applications in the theory of parametric variational systems. After a motivational discussion on the case of separate variables in the definition of the composition map, we introduce a local composition stability notion which generalizes the corresponding local sum stability studied in [8]. On this basis we investigate the metric regularity and the Aubin property of the solution mapping associated to a very general type of parametric variational system. We point out the boundedness constants for the regularity moduli, extending the previous results in the field (see [1], [2], [9]).

## 2 Preliminaries

This section contains some basic definitions and results used in the sequel. In what follows, we suppose that the involved spaces are metric spaces, unless otherwise stated. In this setting,  $B(x, r)$  and  $D(x, r)$  denote the open and the closed ball with center  $x$  and radius  $r$ , respectively. On a product space we take the additive metric. If  $x \in X$  and  $A \subset X$ , one defines the distance from  $x$  to  $A$  as  $d(x, A) := \inf\{d(x, a) \mid a \in A\}$ . As usual, we use the convention  $d(x, \emptyset) = \infty$ . The excess from a set  $A$  to a set  $B$  is defined as  $e(A, B) := \sup\{d(a, B) \mid a \in A\}$ . For a non-empty set  $A \subset X$  we put  $\text{cl } A$  for its topological closure. One says that a set  $A$  is locally complete (closed) if there exists  $r > 0$  such that  $A \cap D(x, r)$  is complete (closed).

Let  $F : X \rightrightarrows Y$  be a multifunction. The domain and the graph of  $F$  are denoted respectively by  $\text{Dom } F := \{x \in X \mid F(x) \neq \emptyset\}$  and  $\text{Gr } F := \{(x, y) \in X \times Y \mid y \in F(x)\}$ . If  $A \subset X$  then  $F(A) := \bigcup_{x \in A} F(x)$ . The inverse set-valued map of  $F$  is  $F^{-1} : Y \rightrightarrows X$  given by  $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ . If  $F_1 : X \rightrightarrows Y, F_2 : X \rightrightarrows Z$ , we define the set-valued map  $(F_1, F_2) : X \rightrightarrows Y \times Z$  by  $(F_1, F_2)(x) := F_1(x) \times F_2(x)$ . For a parametric multifunction  $F : X \times P \rightrightarrows Y$ , we use the notations:  $F_p(\cdot) := F(\cdot, p)$  and  $F_x(\cdot) := F(x, \cdot)$ .

We recall now the concepts of openness at linear rate, metric regularity and Aubin property of a multifunction around the reference point.

**Definition 2.1** *Let  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ .*

*(i)  $F$  is said to be open at linear rate  $L > 0$  around  $(\bar{x}, \bar{y})$  if there exist a positive number  $\varepsilon > 0$  and two neighborhoods  $U \in \mathcal{V}(\bar{x}), V \in \mathcal{V}(\bar{y})$  such that, for every  $\rho \in (0, \varepsilon)$  and every  $(x, y) \in \text{Gr } F \cap [U \times V]$ ,*

$$B(y, \rho L) \subset F(B(x, \rho)). \quad (2.1)$$

*The supremum of  $L > 0$  over all the combinations  $(L, U, V, \varepsilon)$  for which (2.1) holds is denoted by  $\text{lop } F(\bar{x}, \bar{y})$  and is called the exact linear openness bound, or the exact covering bound of  $F$  around  $(\bar{x}, \bar{y})$ .*

(ii)  $F$  is said to have the Aubin property (or to be Lipschitz-like) around  $(\bar{x}, \bar{y})$  with constant  $L > 0$  if there exist two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $x, u \in U$ ,

$$e(F(x) \cap V, F(u)) \leq Ld(x, u). \quad (2.2)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V)$  for which (2.2) holds is denoted by  $\text{lip } F(\bar{x}, \bar{y})$  and is called the exact Lipschitz bound of  $F$  around  $(\bar{x}, \bar{y})$ .

(iii)  $F$  is said to be metrically regular around  $(\bar{x}, \bar{y})$  with constant  $L > 0$  if there exist two neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{y})$  such that, for every  $(x, y) \in U \times V$ ,

$$d(x, F^{-1}(y)) \leq Ld(y, F(x)). \quad (2.3)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V)$  for which (2.3) holds is denoted by  $\text{reg } F(\bar{x}, \bar{y})$  and is called the exact regularity bound of  $F$  around  $(\bar{x}, \bar{y})$ .

The links between the previous notions are as follows (see, e.g., [16, Theorem 9.43], [14, Theorems 1.52]).

**Theorem 2.2** *Let  $F : X \rightrightarrows Y$  be a multifunction and  $(\bar{x}, \bar{y}) \in \text{Gr } F$ . Then  $F$  is open at linear rate around  $(\bar{x}, \bar{y})$  iff  $F^{-1}$  has the Aubin property around  $(\bar{y}, \bar{x})$  iff  $F$  is metrically regular around  $(\bar{x}, \bar{y})$ . Moreover, in every of the previous situations,*

$$(\text{lop } F(\bar{x}, \bar{y}))^{-1} = \text{lip } F^{-1}(\bar{y}, \bar{x}) = \text{reg } F(\bar{x}, \bar{y}).$$

In the case of parametric set-valued maps one has the following partial notions of linear openness, metric regularity and Aubin property around the reference point.

**Definition 2.3** *Let  $F : X \times P \rightrightarrows Y$  be a multifunction and  $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } F$ .*

(i)  $F$  is said to be open at linear rate  $L > 0$  with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  if there exist a positive number  $\varepsilon > 0$  and some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $\rho \in (0, \varepsilon)$ , every  $p \in V$  and every  $(x, y) \in \text{Gr } F_p \cap [U \times W]$ ,

$$B(y, \rho L) \subset F_p(B(x, \rho)). \quad (2.4)$$

The supremum of  $L > 0$  over all the combinations  $(L, U, V, W, \varepsilon)$  for which (2.4) holds is denoted by  $\widehat{\text{lop}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact linear openness bound, or the exact covering bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .

(ii)  $F$  is said to have the Aubin property (or to be Lipschitz-like) with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  with constant  $L > 0$  if there exist some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $x, u \in U$  and every  $p \in V$ ,

$$e(F_p(x) \cap W, F_p(u)) \leq Ld(x, u). \quad (2.5)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V, W)$  for which (2.5) holds is denoted by  $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact Lipschitz bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .

(iii)  $F$  is said to be metrically regular with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{y})$  with constant  $L > 0$  if there exist some neighborhoods  $U \in \mathcal{V}(\bar{x})$ ,  $V \in \mathcal{V}(\bar{p})$ ,  $W \in \mathcal{V}(\bar{y})$  such that, for every  $(x, p, y) \in U \times V \times W$ ,

$$d(x, F_p^{-1}(y)) \leq Ld(y, F_p(x)). \quad (2.6)$$

The infimum of  $L > 0$  over all the combinations  $(L, U, V, W)$  for which (2.6) holds is denoted by  $\widehat{\text{reg}}_x F((\bar{x}, \bar{p}), \bar{y})$  and is called the exact regularity bound of  $F$  in  $x$  around  $((\bar{x}, \bar{p}), \bar{y})$ .

Interchanging the roles of  $p$  and  $x$  one gets a similar set of concepts.

### 3 Linear openness of compositions

This section is devoted to the main result of the paper, i.e. a chain rule for linear openness of set-valued maps. In fact, the results in this section are metric extensions of some previous assertions proved, on Banach spaces, in [8], [9].

The starting point is an implicit multifunction theorem given (with some extra conclusions) in [9, Theorem 3.6]. Let us remark that different versions of this lemma are done in [2, Theorem 3.5] (with functions instead of multifunctions) and in [12, Lemma 2]. It worth to be mentioned that in [9] this conclusion is obtained as a consequence of a more general implicit mapping result on Banach spaces. Here we present the full (direct) proof, on metric spaces, for the reader's convenience.

**Lemma 3.1** *Let  $Y, Z, W$  be metric spaces,  $G : Y \times Z \rightrightarrows W$  be a multifunction and  $(\bar{y}, \bar{z}, \bar{w}) \in Y \times Z \times W$  be such that  $\bar{w} \in G(\bar{y}, \bar{z})$ . Consider next the implicit multifunction  $\Gamma : Z \times W \rightrightarrows Y$  defined by*

$$\Gamma(z, w) := \{y \in Y \mid w \in G(y, z)\}.$$

*Suppose that the following conditions are satisfied:*

*(i)  $G$  has the Aubin property with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $D > 0$ ;*

*(ii)  $G$  is open at linear rate with respect to  $y$  uniformly in  $z$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $C > 0$ .*

*Then there exists  $\gamma > 0$  such that, for every  $(z, w), (z', w') \in D(\bar{z}, \gamma) \times D(\bar{w}, \gamma)$ ,*

$$e(\Gamma(z, w) \cap D(\bar{y}, \gamma), \Gamma(z', w')) \leq \frac{1}{C}(Dd(z, z') + d(w, w')). \quad (3.1)$$

**Proof.** The condition (i) allows us to find  $\alpha > 0$  such that, for every  $y \in B(\bar{y}, \alpha)$  and every  $z, z' \in B(\bar{z}, \alpha)$ ,

$$e(G(y, z) \cap B(\bar{w}, \alpha), G(y, z')) \leq Dd(z, z'). \quad (3.2)$$

Also, from (ii), one can choose  $\alpha > 0$  from before sufficiently small and  $\varepsilon > 0$  such that for every  $\rho \in (0, \varepsilon)$ , every  $z \in B(\bar{z}, \alpha)$  and every  $(y, w) \in \text{Gr } G_z \cap [B(\bar{y}, \alpha) \times B(\bar{w}, \alpha)]$ ,

$$B(w, C\rho) \subset G_z(B(y, \rho)). \quad (3.3)$$

Take  $\gamma > 0$  such that  $\gamma < \frac{\alpha}{2}$ ,  $\gamma < \frac{\alpha}{8D}$ ,  $\frac{1+\gamma}{C}(3\gamma + D\gamma) < \varepsilon$ , and pick arbitrary  $(z, w), (z', w') \in D(\bar{z}, \gamma) \times D(\bar{w}, \gamma)$ . Moreover, consider  $y \in \Gamma(z, w) \cap D(\bar{y}, \gamma)$ . Then  $w \in G(y, z) \cap B(\bar{w}, \frac{\alpha}{2})$ , whence, by (3.2),

$$d(w, G(y, z')) \leq Dd(z, z').$$

Consequently, for every  $\theta \in (0, \min\{\gamma, \frac{\alpha}{4}\})$ , there exists  $w'' \in G(y, z')$  such that

$$d(w, w'') < d(w, G(y, z')) + \theta \leq Dd(z, z') + \theta < \frac{\alpha}{2}.$$

Therefore,  $w'' \in B(\bar{w}, \alpha)$ . If  $w'' = w'$ , then  $y \in \Gamma(z', w')$  and  $d(y, \Gamma(z', w')) = 0$ . Suppose next that  $w'' \neq w'$ . Then  $w' \in B(w'', (1 + \theta)d(w', w''))$ . Take now  $\rho_0 := \frac{1+\theta}{C}d(w', w'') < \frac{1+\gamma}{C}(3\gamma + D\gamma) < \varepsilon$  and apply (3.3) for  $z', y, w''$  and  $\rho_0$ , to get

$$w' \in B(w'', (1 + \theta)d(w', w'')) \subset G(B(y, \rho_0), z').$$

Hence, there exists  $y' \in \Gamma(z', w')$  with  $d(y, y') < \frac{1+\theta}{C}d(w', w'')$ . Then

$$\begin{aligned} d(y, \Gamma(z', w')) &\leq d(y, y') < \frac{1+\theta}{C}d(w', w'') \\ &\leq \frac{1+\theta}{C}(d(w', w) + d(w, w'')) \\ &\leq \frac{1+\theta}{C}(d(w', w) + Dd(z, z') + \theta). \end{aligned}$$

Because  $y$  was arbitrary chosen from  $\Gamma(z, w) \cap D(\bar{y}, \gamma)$ , and also,  $\theta$  can be made arbitrary small, taking the supremum for all such  $y$  and making  $\theta \rightarrow 0$  in the above relation, one obtains (3.1).  $\square$

We are now able to formulate and to prove our main theorem, a linear openness result for a fairly general set-valued composition. This result was proved on Banach spaces in [8], following the usual technique offered by Ekeland Variational Principle (see the comments in Section 1). Now, we follow the "dual" approach, by means of iteration procedure of Lyusternik type. Remark that we have here a purely metric result, without any linear structure, which will allow us later to clearly emphasize the role of shift-invariance property of the metric in the set-valued version of Lyusternik-Graves Theorem.

**Theorem 3.2** *Let  $X, Y, Z, W$  be metric spaces,  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \rightrightarrows Z$  and  $G : Y \times Z \rightrightarrows W$  be multifunctions and  $(\bar{x}, \bar{y}, \bar{z}, \bar{w}) \in X \times Y \times Z \times W$  such that  $(\bar{x}, \bar{y}) \in \text{Gr } F_1$ ,  $(\bar{x}, \bar{z}) \in \text{Gr } F_2$  and  $((\bar{y}, \bar{z}), \bar{w}) \in \text{Gr } G$ . Let  $H : X \rightrightarrows W$  be given by*

$$H(x) := G(F_1(x), F_2(x)) \text{ for every } x \in X.$$

*Suppose that the following assumptions are satisfied:*

- (i)  $\text{Gr } F_1$ ,  $\text{Gr } F_2$  are locally complete around  $(\bar{x}, \bar{y})$ ,  $(\bar{x}, \bar{z})$ , respectively, and  $\text{Gr } G$  is locally closed around  $((\bar{y}, \bar{z}), \bar{w})$ ;
- (ii)  $F_1$  is open at linear rate  $L > 0$  around  $(\bar{x}, \bar{y})$ ;
- (iii)  $F_2$  has the Aubin property around  $(\bar{x}, \bar{z})$  with constant  $M > 0$ ;
- (iv)  $G$  is open at linear rate with respect to  $y$  uniformly in  $z$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $C > 0$ ;
- (v)  $G$  has the Aubin property with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $D > 0$ ;
- (vi)  $LC - MD > 0$ .

*Then there exists  $\varepsilon > 0$  such that, for every  $\rho \in (0, \varepsilon)$ ,*

$$B(\bar{w}, (LC - MD)\rho) \subset H(B(\bar{x}, \rho)).$$

*Moreover, there exists  $\varepsilon' > 0$  such that, for every  $\rho \in (0, \varepsilon')$  and every  $(x', y', z', w') \in B(\bar{x}, \varepsilon') \times B(\bar{y}, \varepsilon') \times B(\bar{z}, \varepsilon') \times B(\bar{w}, \varepsilon')$  such that  $(y', z') \in (F_1, F_2)(x')$  and  $w' \in G(y', z')$ ,*

$$B(w', (LC - MD)\rho) \subset H(B(x', \rho)).$$

**Proof.** Without loosing the generality, the assumptions made upon the involved mappings yield the existence of  $\alpha > 0$  such that

1.  $\text{Gr } F_1 \cap [D(\bar{x}, \alpha) \times D(\bar{y}, \alpha)]$  and  $\text{Gr } F_2 \cap [D(\bar{x}, \alpha) \times D(\bar{z}, \alpha)]$  are complete, and  $\text{Gr } G \cap [D(\bar{y}, \alpha) \times D(\bar{z}, \alpha) \times D(\bar{w}, \alpha)]$  is closed.

2. for every  $(x, y) \in B(\bar{x}, \alpha) \times B(\bar{y}, \alpha)$ ,

$$d(x, F_1^{-1}(y)) \leq \frac{1}{L}d(y, F_1(x)) \quad (3.4)$$

3. for every  $x, x' \in B(\bar{x}, \alpha)$ ,

$$e(F_2(x) \cap B(\bar{z}, \alpha), F_2(x')) \leq Md(x, x'). \quad (3.5)$$

4. for every  $(z, w), (z', w') \in B(\bar{z}, \alpha) \times B(\bar{w}, \alpha)$ ,

$$e(\Gamma(z, w) \cap D(\bar{y}, \alpha), \Gamma(z', w')) \leq \frac{1}{C}(Dd(z, z') + d(w, w')), \quad (3.6)$$

where  $\Gamma$  is defined in Lemma 3.1.

5. for every  $y \in B(\bar{y}, \alpha)$  and every  $z, z' \in B(\bar{z}, \alpha)$ ,

$$e(G(y, z) \cap B(\bar{w}, \alpha), G(y, z')) \leq Dd(z, z'). \quad (3.7)$$

Choose  $\varepsilon > 0$  such that

$$\begin{aligned} \varepsilon &< \alpha \\ L\varepsilon &< \alpha \\ M\varepsilon &< \alpha \\ (LC - MD)\varepsilon &< \alpha \\ C^{-1}(1 + \varepsilon)(LC - MD)\varepsilon &< \alpha \\ (LC)^{-1}(1 + 2\varepsilon)(LC - MD)\varepsilon &< \alpha \\ M(LC)^{-1}(1 + 3\varepsilon)(LC - MD)\varepsilon &< \alpha \\ MD(LC)^{-1}(1 + 4\varepsilon) &< 1 \\ \left(1 + (1 + 4\varepsilon)\frac{MD}{LC}\right)(LC - MD)\varepsilon &< \alpha \end{aligned} \quad (3.8)$$

and fix  $\rho \in (0, \varepsilon)$ . Take now  $w \in B(\bar{w}, (LC - MD)\rho)$ ,  $w \neq \bar{w}$ . Then, there exists  $\delta \in (0, \varepsilon)$  such that

$$d(w, \bar{w}) < \left(\frac{LC}{1 + 4\delta} - MD\right)\rho < \left(\frac{LC}{1 + 2\delta} - MD\right)\rho < (LC - MD)\rho. \quad (3.9)$$

Define  $x_0 := \bar{x}$ ,  $y_0 := \bar{y}$ ,  $z_0 := \bar{z}$ ,  $w_0 := \bar{w}$ . Using (3.6) with  $z_0$  instead of  $z$  and  $z'$ ,  $w_0$  instead of  $w$  and  $w \in B(\bar{w}, \alpha)$  instead of  $w'$ , one has that

$$d(y_0, \Gamma(z_0, w)) \leq e(\Gamma(z_0, w_0) \cap B(\bar{y}, \alpha), \Gamma(z_0, w)) < \frac{1 + \delta}{C}d(w_0, w)$$

(where  $\delta > 0$  is the one chosen before; notice that  $w \neq w_0$ ), so there exists  $y_1 \in \Gamma(z_0, w)$  such that

$$d(y_1, y_0) < \frac{1 + \delta}{C}d(w_0, w) < \frac{1 + \varepsilon}{C}(LC - MD)\varepsilon < \alpha. \quad (3.10)$$

Hence,  $y_1 \in B(\bar{y}, \alpha)$ . One can use now (3.4) for  $(x_0, y_1)$  instead of  $(x, y)$  and then

$$d(x_0, F_1^{-1}(y_1)) \leq \frac{1}{L}d(y_1, F_1(x_0)).$$



Because  $w \neq \bar{w}$ , there exists  $x_1 \in F_1^{-1}(y_1)$  such that

$$\begin{aligned} d(x_1, x_0) &< d(x_0, F_1^{-1}(y_1)) + \frac{\delta}{LC}d(w_0, w) \leq \frac{1}{L}d(y_1, F_1(x_0)) + \frac{\delta}{LC}d(w_0, w) \\ &\leq \frac{1}{L}d(y_1, y_0) + \frac{\delta}{LC}d(w_0, w) < \frac{1+2\varepsilon}{LC}(LC - MD)\varepsilon < \alpha. \end{aligned} \quad (3.11)$$

Consequently,  $x_1 \in B(\bar{x}, \alpha)$ . Now, one can use (3.5) for  $x_1, x_0$  instead of  $x, x'$  to get that

$$d(z_0, F_2(x_1)) \leq e(F_2(x_0) \cap B(\bar{z}, \alpha), F_2(x_1)) \leq Md(x_1, x_0),$$

so there exists  $z_1 \in F_2(x_1)$  such that

$$\begin{aligned} d(z_1, z_0) &< d(z_0, F_2(x_1)) + \frac{\delta M}{LC}d(w_0, w) \\ &\leq Md(x_1, x_0) + \frac{\delta M}{LC}d(w_0, w) < M\frac{1+3\varepsilon}{LC}(LC - MD)\varepsilon < \alpha. \end{aligned} \quad (3.12)$$

Hence,  $(y_1, z_1) \in (F_1, F_2)(x_1) \cap [B(\bar{y}, \alpha) \times B(\bar{z}, \alpha)]$ .

Finally, one can use (3.7) to have that

$$e(G(y_1, z_0) \cap B(\bar{w}, \alpha), G(y_1, z_1)) \leq Dd(z_1, z_0).$$

Consequently, because  $w \in G(y_1, z_0) \cap B(\bar{w}, \alpha)$ , there exists  $w_1 \in G(y_1, z_1)$  such that

$$d(w_1, w) < d(w, G(y_1, z_1)) + \frac{\delta MD}{LC}d(w_0, w) \leq Dd(z_1, z_0) + \frac{\delta MD}{LC}d(w_0, w). \quad (3.13)$$

Moreover, remark that  $w_1 \in H(x_1)$ . Then,

$$\begin{aligned} d(w_1, w) &< Dd(z_1, z_0) + \frac{\delta MD}{LC}d(w_0, w) < MDd(x_1, x_0) + \frac{2\delta MD}{LC}d(w_0, w) \\ &< MD \left( \frac{1}{L}d(y_1, y_0) + \frac{\delta}{LC}d(w_0, w) \right) + \frac{2\delta MD}{LC}d(w_0, w) \leq \frac{MD}{LC}(1+4\delta)d(w_0, w) = Kd(w_0, w), \end{aligned} \quad (3.14)$$

where  $K := \frac{MD}{LC}(1+4\delta) < \frac{MD}{LC}(1+4\varepsilon) < 1$ . Also,

$$\begin{aligned} d(w_1, \bar{w}) &\leq d(w_1, w) + d(w, w_0) \\ &\leq (1+K)(LC - MD)\rho < \left( 1 + (1+4\varepsilon)\frac{MD}{LC} \right) (LC - MD)\varepsilon < \alpha. \end{aligned} \quad (3.15)$$

If  $w_1 = w$ , then  $w \in H(x_1)$ , with

$$\begin{aligned} d(x_1, \bar{x}) &\leq \frac{1}{LC}(1+2\delta)d(w_0, w) < \frac{1}{LC}(1+2\delta) \left( \frac{LC}{1+2\delta} - MD \right) \rho \\ &= \left( 1 - (1+2\delta)\frac{MD}{LC} \right) \rho < \rho. \end{aligned}$$

Hence,  $w \in H(B(\bar{x}, \rho))$  and the proof is finished. Suppose next that  $w_1 \neq w$ .



Now, we intend to construct the sequences  $x_n, y_n, z_n, w_n$  such that, for  $n = 0, 1, 2, \dots$ , one has  $w_n \neq w$  and

$$\begin{aligned} y_{n+1} &\in \Gamma(z_n, w) \cap B(\bar{y}, \alpha) \text{ is such that } d(y_{n+1}, y_n) < \frac{1+\delta}{C}d(w_n, w). \\ x_{n+1} &\in F_1^{-1}(y_{n+1}) \cap B(\bar{x}, \alpha) \text{ is such that } d(x_{n+1}, x_n) < \frac{1}{L}d(y_{n+1}, y_n) + \frac{\delta}{LC}d(w_n, w) \\ z_{n+1} &\in F_2(x_{n+1}) \cap B(\bar{z}, \alpha) \text{ is such that } d(z_{n+1}, z_n) < Md(x_{n+1}, x_n) + \frac{\delta M}{LC}d(w_n, w), \\ w_{n+1} &\in G(y_{n+1}, z_{n+1}) \cap B(\bar{w}, \alpha) \text{ is such that } d(w_{n+1}, w) < Dd(z_{n+1}, z_n) + \frac{\delta MD}{LC}d(w_n, w). \end{aligned} \quad (3.16)$$

Observe that, in view of (3.10), (3.11), (3.12), (3.13), (3.15), for  $n = 0$ , all the assertions from the previous formula are satisfied by  $x_1, y_1, z_1, w_1$ . Suppose that for some  $p \geq 1$  we have generated  $x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p, z_1, z_2, \dots, z_p$  and  $w_1, w_2, \dots, w_p$  satisfying (3.16).

Then

$$d(w_p, w) \leq Kd(w_{p-1}, w) \leq \dots \leq K^p d(w_0, w),$$

whence

$$\begin{aligned} d(w_p, \bar{w}) &\leq d(w_p, w) + d(w, \bar{w}) \leq (1 + K^p)(LC - MD)\varepsilon \\ &\leq (1 + K)(LC - MD)\varepsilon < (1 + (1 + 4\varepsilon)\frac{MD}{LC})(LC - MD)\varepsilon < \alpha. \end{aligned} \quad (3.17)$$

If  $w_p = w$ , then  $w \in H(x_p)$ . Also, combining the inequalities from (3.16), we have

$$\begin{aligned} d(x_p, \bar{x}) &\leq \sum_{i=1}^p d(x_i, x_{i-1}) \leq \frac{1+2\delta}{LC} \sum_{i=0}^{p-1} d(w_i, w) \\ &\leq \frac{1+4\delta}{LC} \sum_{i=0}^{p-1} K^i d(w_0, w) \leq \frac{1+4\delta}{LC} \cdot \frac{1-K^p}{1-K} d(w_0, w) \\ &< \frac{1+4\delta}{LC \left(1 - (1+4\delta)\frac{DM}{LC}\right)} \left(\frac{LC}{1+4\delta} - DM\right) \rho = \rho. \end{aligned} \quad (3.18)$$

Hence,  $x_p \in B(\bar{x}, \rho)$  and the proof is finished. Suppose next that  $w_p \neq w$ .

Because  $w_p, w \in B(\bar{w}, \alpha)$ ,  $y_p \in \Gamma(z_p, w_p) \cap B(\bar{y}, \alpha)$ , one can find, using again (3.6),  $y_{p+1} \in \Gamma(z_p, w)$  such that

$$d(y_{p+1}, y_p) < \frac{1+\delta}{C}d(w_p, w). \quad (3.19)$$

Also,

$$\begin{aligned} d(y_{p+1}, \bar{y}) &\leq \sum_{i=1}^{p+1} d(y_i, y_{i-1}) \leq \frac{1+\delta}{C} \sum_{i=0}^p d(w_i, w) \\ &\leq \frac{1+4\delta}{C} \sum_{i=0}^p K^i d(w_0, w) \leq \frac{1+4\delta}{C} \cdot \frac{1-K^{p+1}}{1-K} d(w_0, w) \\ &\leq \frac{1+4\delta}{C(1-K)} \left(\frac{LC}{1+4\delta} - MD\right) \varepsilon = L\varepsilon < \alpha. \end{aligned} \quad (3.20)$$

Hence,  $y_{p+1} \in B(\bar{y}, \alpha)$ . Then, one can use (3.4) for  $(x_p, y_{p+1})$  instead of  $(x, y)$  to have

$$d(x_p, F_1^{-1}(y_{p+1})) \leq L^{-1}d(y_{p+1}, F_1(x_p)).$$

Again, we are in the case  $w \neq w_p$ , so there exists  $x_{p+1} \in F_1^{-1}(y_{p+1})$  such that

$$\begin{aligned} d(x_{p+1}, x_p) &< d(x_p, F_1^{-1}(y_{p+1})) + \frac{\delta}{LC}d(w_p, w) \leq \frac{1}{L}d(y_{p+1}, F_1(x_p)) + \frac{\delta}{LC}d(w_p, w) \\ &\leq \frac{1}{L}d(y_{p+1}, y_p) + \frac{\delta}{LC}d(w_p, w). \end{aligned} \quad (3.21)$$

As above, one has

$$d(x_{p+1}, \bar{x}) < \rho < \varepsilon < \alpha.$$

Hence, one can apply (3.5) for  $x_{p+1}, x_p$  instead of  $x, x'$  to get  $z_{p+1} \in F_2(x_{p+1})$  such that

$$d(z_{p+1}, z_p) < Md(x_{p+1}, x_p) + \frac{\delta M}{LC}d(w_p, w). \quad (3.22)$$

But, using (3.18),

$$\begin{aligned} d(z_{p+1}, \bar{z}) &\leq \sum_{i=1}^{p+1} d(z_i, z_{i-1}) \leq M \sum_{i=1}^{p+1} d(x_i, x_{i-1}) + \frac{\delta M}{LC} \sum_{i=1}^{p+1} d(w_{i-1}, w) \\ &\leq \frac{(1+4\delta)M}{LC} \sum_{i=0}^p d(w_i, w) \leq M\rho < M\varepsilon < \alpha. \end{aligned} \quad (3.23)$$

So,  $z_{p+1} \in B(\bar{z}, \alpha)$ , whence one can use (3.7) to get  $w_{p+1} \in G(y_{p+1}, z_{p+1})$  such that

$$d(w_{p+1}, w) < Dd(z_{p+1}, z_p) + \frac{\delta MD}{LC}d(w_p, w). \quad (3.24)$$

Finally, one can prove like in (3.17) that  $d(w_{p+1}, \bar{w}) < \alpha$ . Remark also that  $w_{p+1} \in H(x_{p+1})$ .

In this moment, we have completely finished the induction step, hence (3.16) holds for every positive integer  $n$ .

We intend to prove next that the sequences  $(x_n), (y_n), (z_n)$  satisfy the Cauchy condition. For this, observe first that, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} d(w_{n+1}, w) &< Dd(z_{n+1}, z_n) + \frac{\delta MD}{LC}d(w_n, w) < MDd(x_{n+1}, x_n) + \frac{2\delta MD}{LC}d(w_n, w) \\ &< MD \left( \frac{1}{L}d(y_{n+1}, y_n) + \frac{\delta}{LC}d(w_n, w) \right) + \frac{2\delta MD}{LC}d(w_n, w) \leq Kd(w_n, w) \leq K^n d(w_0, w), \end{aligned} \quad (3.25)$$

hence  $w_n \rightarrow w$  (because  $K < 1$ ).

Also, for every  $p \in \mathbb{N}$ ,

$$\begin{aligned} d(x_{n+p}, x_n) &\leq \sum_{i=1}^p d(x_{n+i}, x_{n+i-1}) \leq \frac{1+2\delta}{LC} \sum_{i=1}^p d(w_{n+i-1}, w) \\ &\leq \frac{1+2\delta}{LC} \sum_{i=1}^p K^{n+i-2} d(w_0, w) \leq \frac{1+2\varepsilon}{LC} \cdot \frac{K^{n-1}}{1-K} d(w_0, w), \end{aligned}$$

so, for  $n$  sufficiently large,  $d(x_{n+p}, x_n)$  can be made arbitrary small. Similar assertions hold for  $(y_n)$  and  $(z_n)$ , which can be proven in the line of (3.20) and (3.23). Because  $(x_n, y_n, z_n) \in \text{Gr}(F_1, F_2) \cap [D(\bar{x}, \alpha) \times D(\bar{y}, \alpha) \times D(\bar{z}, \alpha)]$  for every  $n \in \mathbb{N}$ , one can find, using 1., that there exist  $(x, y, z) \in \text{Gr}(F_1, F_2)$  such that  $(x_n, y_n, z_n) \rightarrow (x, y, z)$ . Also, because  $\text{Gr } G \ni (y_n, z_n, w_n) \rightarrow (y, z, w)$  and  $\text{Gr } G$  is closed, we obtain that  $w \in G(y, z)$ , hence  $w \in H(x)$ . To complete the proof, it remains to prove that  $d(x, \bar{x}) < \rho$ .

For this, taking into account (3.18) and (3.9), observe that

$$\begin{aligned} d(x, \bar{x}) &\leq d(x, x_n) + d(x_n, \bar{x}) \leq d(x, x_n) + \frac{1+2\delta}{LC} \cdot \frac{1-K^n}{1-K} d(w_0, w) \\ &\leq d(x, x_n) + \frac{1+2\delta}{LC \left(1 - (1+4\delta) \frac{DM}{LC}\right)} \left(\frac{LC}{1+4\delta} - DM\right) \rho \\ &= d(x, x_n) + \frac{1+2\delta}{1+4\delta} \rho \\ &= d(x, x_n) + \rho - \frac{2\delta}{1+4\delta} \rho. \end{aligned}$$

Since  $x_n \rightarrow x$ , for  $n$  sufficiently large,  $d(x, x_n) - \frac{2\delta}{1+4\delta} \rho < 0$ , whence  $d(x, \bar{x}) < \rho$  and the proof of the first part is done.

For the second part, take  $\varepsilon > 0$  such that all the inequalities from (3.8) are satisfied with  $\alpha$  replaced by  $\frac{\alpha}{2}$ . Furthermore, define  $\varepsilon' := \frac{\varepsilon}{2}$  and pick  $(x', y', z', w') \in B(\bar{x}, \varepsilon') \times B(\bar{y}, \varepsilon') \times B(\bar{z}, \varepsilon') \times B(\bar{w}, \varepsilon')$  such that  $(y', z') \in (F_1, F_2)(x')$ ,  $w' \in G(y', z')$ . Also, choose  $w \in B(w', (LC - MD)\rho)$ . Then  $B(x', \frac{\varepsilon}{2}) \subset B(\bar{x}, \varepsilon) \subset B(\bar{x}, \alpha)$ , and similar assertions hold for the other balls. Then the proof becomes very similar to the one of the first part, starting with  $x_0 := x', y_0 := y', z_0 := z', w_0 := w'$ .  $\square$

We remark that the proof based on iteration procedure allows to use more refined completeness conditions on the initial data (compare with [8, Theorem 3.3]). The next sections are two fold: firstly, they illustrate the main result, Theorem 3.2, by some direct consequences and, secondly, some applications to parametric variational systems are derived.

## 4 Consequences

We start the discussion on the consequences of Theorem 3.2 by the study of two particular cases of set-valued maps  $G$  which fulfill the properties required in that theorem.

**Remark 4.1** *First, we consider the situation where  $Y = Z = W$  is a linear metric space endowed with a shift-invariant metric  $d$  and  $G : Y \times Y \rightrightarrows Y$  is defined by  $G(y, z) = \{y - z\}$ . Take  $\bar{y}, \bar{z} \in Y$  and  $\bar{w} = \bar{y} - \bar{z}$ . To show that  $G$  has the Aubin property with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $D = 1$  is an easy task because for every  $\varepsilon > 0$ ,  $y \in B(\bar{y}, \varepsilon)$ ,  $z_1, z_2 \in B(\bar{z}, \varepsilon)$ ,*

$$d(y - z_1, y - z_2) = d(z_1, z_2).$$

*Similarly, in order to show that  $G$  is open at linear rate with respect to  $y$  uniformly in  $z$  around  $((\bar{y}, \bar{z}), \bar{w})$ , with constant  $C = 1$ , consider the next remark. Take  $\rho > 0$  and  $w \in B(\bar{w}, \rho)$ . Hence*

$$d(w, \bar{y} - \bar{z}) < \rho$$

and taking again into account the shift-invariance of  $d$ ,

$$d(\bar{y}, w + \bar{z}) < \rho,$$

whence  $w + \bar{z} \in B(\bar{y}, \rho)$ , i.e.  $w \in B(\bar{y}, \rho) - \bar{z} = G_{\bar{z}}(B(\bar{y}, \rho))$ . The linearity ensures the announced property.

Next, we would like to emphasize that if one drops the shift-invariance of the metric  $d$ , then the properties of  $G$  cannot be guaranteed.

**Example 4.2** Consider, for instance,  $Y = \mathbb{R}$  (the real line) with the metric  $d(x, y) = |x^3 - y^3|$ . This metric does not fulfill the shift-invariance property, and  $G$  is not open at linear rate around any  $((\bar{y}, \bar{z}), \bar{w})$ . Indeed, with the above notations let us take  $\bar{y} = \bar{z} = 1$ . If  $G$  would satisfy the openness property around  $(1, 1, 0)$ , then for any  $\rho > 0$  small enough, the inequality

$$|w^3| < C\rho \Leftrightarrow |w| < \sqrt[3]{C\rho}$$

should imply

$$\begin{aligned} |(w+1)^3 - 1| &< \rho \Leftrightarrow \\ |w^3 + 3w^2 + 3w| &= |w|(w^2 + 3w + 3) < \rho. \end{aligned}$$

Take  $\rho \in (n^{-1}, 2n^{-1})$  for  $n \in \mathbb{N}$  large enough and  $w_n := \sqrt[3]{Cn^{-1}}$ . Then one should have

$$|w_n^3 + 3w_n^2 + 3w_n| < 2n^{-1}$$

i.e.

$$Cn^{-1} + 3\sqrt[3]{C^2n^{-2}} + 3\sqrt[3]{Cn^{-1}} < 2n^{-1}$$

for all  $n$  large enough, which is not possible.

Nevertheless, for some metrics without shift-invariance,  $G$  could fulfill the respective properties, but the constants  $C$  and  $D$  would depend both on the metric and the point  $(\bar{y}, \bar{z})$ . Let us consider an example of this type. Take  $Y = \mathbb{R}$  with the metric

$$d(x, y) = \begin{cases} 2|x - y| & \text{if } x, y \leq 0 \\ |x - y| & \text{if } x, y > 0 \\ |2x - y| & \text{if } x \leq 0, y > 0 \\ |x - 2y| & \text{if } x > 0, y \leq 0. \end{cases}$$

Obviously, this is not a shift-invariance metric. Take  $\bar{y} = 1$ ,  $\bar{z} = 2$ . Then  $\bar{w} = -1$ . It is easy to see that  $G$  satisfy the needed properties for  $C = 2$  and  $D = 2$  because taking small balls one works in the left-hand sides with negative numbers and, in the right-hand sides (after translation) with positive numbers.

A second important remark of this section is as follows.

**Remark 4.3** Take  $Y = Z$  as a metric space,  $W = \mathbb{R}$  and  $G : Y \times Y \rightrightarrows \mathbb{R}$ ,  $G(y, z) = \{d(y, z)\}$ . Take  $\bar{y}, \bar{z} \in Y$  and  $\bar{w} = d(\bar{y}, \bar{z})$ . The fact that  $G$  has the Aubin property with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $D = 1$  is immediate.

Suppose now that  $\bar{w} \neq 0$ , i.e.  $\bar{y} \neq \bar{z}$  and write down how the  $C$ -linear openness property of  $G$  looks like in this case. Denote  $\alpha := d(\bar{y}, \bar{z}) > 0$ . Take  $\varepsilon := 2^{-1}C^{-1}\alpha$ ,  $\rho \in (0, \varepsilon)$ ,  $z \in B(\bar{z}, 4^{-1}\alpha)$ ,

$(y, w) \in \text{Gr } G_z \cap [B(\bar{y}, 4^{-1}\alpha) \times B(\bar{w}, \alpha)]$  and  $\mu \in B(w, C\rho)$ . Then  $d(y, z) \geq 2^{-1}\alpha > C\rho$ , because, otherwise, one gets the contradiction

$$d(\bar{y}, \bar{z}) \leq d(\bar{y}, y) + d(y, z) + d(z, \bar{z}) < \alpha.$$

One has  $|\mu - d(y, z)| < C\rho$ , whence

$$\mu \in (d(y, z) - C\rho, d(y, z) + C\rho).$$

But  $G_z(B(y, \rho)) = \{d(u, z) \mid u \in B(y, \rho)\}$ . Now, in order to have the openness property for  $G$  one needs the following inclusion

$$(d(y, z) - C\rho, d(y, z) + C\rho) \subset \{d(u, z) \mid u \in B(y, \rho)\}. \quad (4.1)$$

Notice that the reverse inclusion always holds for  $C = 1$

There are some particular situations where (4.1) fails. In order to illustrate this, we provide some examples.

**Example 4.4** A very simple case is that of the discrete metric  $d(x, y) = 0$  if  $x = y$  and  $d(x, y) = 1$  otherwise. A more elaborated example is the following one. Take  $Y = \mathbb{R}^2$ , denote by  $d_E$  the usual (Euclidean) distance on  $Y$  and consider the following distance

$$d(x, y) = \begin{cases} d_E(x, y) & \text{if } d_E(x, 0) = d_E(y, 0) \\ d_E(x, 0) + d_E(y, 0) & \text{if } d_E(x, 0) \neq d_E(y, 0). \end{cases}$$

If one takes  $z = (0, 0)$  and  $y = (0, 1)$ , then  $\{d(u, z) \mid u \in B(y, \rho)\} = \{1\}$  for any  $\rho < 1$  since  $B(y, \rho)$  is an arc of the unit circle. Of course, in these conditions, (4.1) fails.

Fortunately, there are as well some general remarkable cases where the inclusion (4.1) holds.

**Example 4.5** Firstly, this is the case when  $Y$  is a normed vector space (endowed with a norm denoted  $\|\cdot\|$ ). Indeed, for any  $a \in (-\rho, \rho)$  consider  $u := y + a \|y - z\|^{-1} (y - z) \in B(y, \rho)$ . Then

$$\|u - z\| = \|y - z\| + a$$

which shows (4.1) whence the second property of  $G$  in Theorem 3.2 holds for  $C = 1$ .

**Example 4.6** Secondly, we emphasize that the Riemann metric on a finite dimensional complete connected Riemannian manifold also satisfies the property in certain situations. In this framework we experience another restriction on the points  $\bar{y}, \bar{z}$  apart from the one we have met before (i.e.  $\bar{y} \neq \bar{z}$ ). More specifically, in the case  $\bar{y} = \bar{z}$  it is not possible, for instance, to cover the negative part of the interval  $(d(\bar{y}, \bar{z}) - C\rho, d(\bar{y}, \bar{z}) + C\rho) = (-C\rho, C\rho)$ . In some cases, on Riemannian manifolds, it is not possible to cover the positive part of this interval. Roughly speaking, in order to emphasize the difficulties which could arise, let us take the case of  $S^2 \subset \mathbb{R}^3$  sphere, where the points  $\bar{y}, \bar{z}$  are antipodal. In this case, it is not possible to get a distance larger than  $d(\bar{y}, \bar{z}) = \pi$ . In fact, it happens that for some points in  $B(\bar{y}, \rho)$ , there are geodesic arcs connecting these points with  $\bar{z}$  which are not minimizing. This suggests that the points  $\bar{y}, \bar{z}$  should be taken close enough.

After this preliminaries, let us recall the main technical fact which allows us to formulate our conclusions. To this end, we make use of several notations and results from [4].

Let  $Y$  be a finite dimensional complete connected Riemannian manifold. Recall that the Riemann metric is given by a 2-tensor field that is symmetric and positively definite. A Riemann metric determines an inner product and a norm on each tangent space  $T_y Y$ , usually written as  $\|\cdot\|_y$ , where the subscript  $y$  may be omitted. For a smooth curve  $\gamma : I \subset \mathbb{R} \rightarrow Y$ , where  $I$  is an interval,  $\gamma'(t) \in T_{\gamma(t)} Y$  for all  $t \in I$  and  $\|\gamma'(\cdot)\| \in C^\infty(I)$ ; then one can define the length of  $\gamma$  on an interval  $[a, b] \subset I$  as

$$l(\gamma)_a^b := \int_a^b \|\gamma'(t)\| dt.$$

Given two points  $y, z \in Y$  one defines the Riemannian distance from  $y$  to  $z$  by,

$$d(y, z) = \inf_{\gamma} l(\gamma)_0^1,$$

where the infimum is taken over all piecewise smooth curves  $\gamma : [0, 1] \rightarrow Y$  connecting  $y$  and  $z$ . Thus  $d$  is a distance and the topology induced by  $d$  coincides with the topology of  $Y$  as a differentiable manifold. Furthermore, the celebrated Hopf-Rinow Theorem (see [4, Theorem 2.8, p. 146]) asserts that  $(Y, d)$  is a complete metric space. Moreover, for every  $y \in Y$  the exponential map  $\exp_p$  is defined on all of  $T_p Y$ . Recall that  $\exp_p(0) = p$  and for  $v \in T_p Y \setminus \{0\}$ ,  $\exp_p(v)$  is a point in  $Y$  obtained by going out the length  $\|v\|$  starting from  $p$  along a geodesic which passes through  $y$  with velocity equal to  $\|v\|^{-1} v$ . A geodesic  $\gamma$  connecting  $y$  and  $z$  is called a minimizing geodesic if its arc-length is equal to the Riemannian distance between  $y$  and  $z$ . A curve  $\gamma : [0, 1] \rightarrow Y$  is a minimizing geodesic connecting  $y$  and  $z$  (i.e.  $\gamma(0) = y$  and  $\gamma(1) = z$ ) if and only if there is  $v \in T_y Y$  such that  $\|v\| = d(y, z)$  and  $\gamma(t) = \exp_p(tv)$  for each  $t \in [0, 1]$ . In view of the equivalence between the topology of the Riemann distance and the original topology of  $Y$ , for every  $y \in Y$  and for every  $\delta$  small enough, the ball  $B(y, \delta)$  with respect to the Riemann distance coincides normal ball (or geodesic ball) of center  $y$  and radius  $\delta$  (see [4, p. 70, 146]).

Let  $\bar{y} \in Y$ . According to [4, Proposition 4.2, p. 76] there exists  $\beta > 0$  such that the geodesic ball, (denoted  $B$ ) of center  $\bar{y}$  and radius  $\beta$  is strongly convex, i.e. any two points in that ball can be joined by a minimizing geodesic.

Now we are ready to pass to the proof of the openness property of the Riemann distance by proving (4.1) with  $C = 1$ , keeping unchanged the notations of Remark 4.3 and taking  $\bar{z} \in B$  different from  $\bar{y}$ . However, we need first to adjust (eventually) the constant  $\varepsilon$  and the neighborhood of  $\bar{y}$ . For this, one applies [4, Theorem 3.7, p. 72] at  $\bar{y}$ . Then there exists  $\theta > 0$  such that  $B(\bar{y}, \theta) \subset B$  is a normal neighborhood for each of its points. Take  $y \in B(\bar{y}, \min\{2^{-1}\theta, 4^{-1}\alpha\})$ . Then  $B(y, 2^{-1}\theta) \subset B(\bar{y}, \theta)$  and  $B(y, 2^{-1}\theta)$  is a normal ball at  $y$ . Consider now  $\varepsilon' = \min\{\varepsilon, 2^{-1}\theta\}$ ,  $\rho \in (0, \varepsilon')$ . It is enough to show that for every  $a \in (0, \rho)$ , there exist  $u_1, u_2 \in B(y, \rho)$  such that  $d(u_1, z) = d(y, z) - a$ , and  $d(u_2, z) = d(y, z) + a$ . Now  $B(y, a)$  is a normal ball for  $y$ . We follow the main idea of proof of Hopf-Rinow Theorem from [4, page 147]. Let  $x_0$  the point on the (compact) boundary of  $B(y, a)$  where the continuous function  $d(z, \cdot)$  attains its minimum. Then there exists  $v \in T_y Y$  with  $\|v\| = 1$  such that  $x_0 = \exp_y(av)$ . Inspecting the proof in [4, page 147] one observes that it is practically shown that for any  $s \in [0, d(y, z)]$ ,

$$d(\exp_y(sv), z) = d(y, z) - s,$$

i.e.  $y, x_0, z$  are on a geodesic. In particular,  $d(\exp_y(av), z) = d(y, z) - a$ . Then we have found  $u_1 = \exp_y(av) \in B(y, \rho)$  having the desired property. Now consider  $u_2 = \exp_y(-av) \in B(y, \rho)$ . Then since we work on a minimizing geodesic (the points are in  $B$ ),  $d(u_2, z) = d(y, z) + d(y, u_2) = d(y, z) + a$  and the thesis is finally proved.

As a first application we deduce a well-known set-valued metric version of Lyusternik-Graves Theorem. In view of Theorem 3.2 and Remark 4.1, the proof is straightforward. Moreover, the necessity of the shift-invariance metric is now clearly emphasized and, taking into account the last remark in Example 4.2, one can even drop this requirement in order to get local metric versions of this result with different constants.

**Theorem 4.7** *Let  $X$  be a metric space,  $Y$  be a linear metric space with shift-invariant metric,  $F_1 : X \rightrightarrows Y$  and  $F_2 : X \rightrightarrows Y$  be multifunctions and  $(\bar{x}, \bar{y}_1, \bar{y}_2) \in X \times Y \times Y$  such that  $(\bar{x}, \bar{y}_1) \in \text{Gr } F_1$  and  $(\bar{x}, \bar{y}_2) \in \text{Gr } F_2$ . Suppose that the following assumptions are satisfied:*

- (i)  $\text{Gr } F_1$  and  $\text{Gr } F_2$  are locally complete around  $(\bar{x}, \bar{y}_1)$  and  $(\bar{x}, \bar{y}_2)$ , respectively;
- (ii)  $F_1$  is metrically regular around  $(\bar{x}, \bar{y}_1)$  with constant  $l > 0$ ;
- (iv)  $F_2$  has the Aubin property  $(\bar{x}, \bar{y}_2)$  with constant  $m > 0$ ;
- (v)  $lm < 1$ .

*Then there exists  $\varepsilon > 0$  such that, for every  $\rho \in (0, \varepsilon)$ ,*

$$B(\bar{y}_1 - \bar{y}_2, (l^{-1} - m)\rho) \subset (F_1 - F_2)(B(\bar{x}, \rho)).$$

*Moreover, there exists  $\varepsilon' > 0$  such that, for every  $\rho \in (0, \varepsilon')$ , and every  $(x', y'_1, y'_2) \in \text{Gr}(F_1, F_2) \cap [B(\bar{x}, \varepsilon) \times B(\bar{y}_1, \varepsilon) \times B(\bar{y}_2, \varepsilon)]$ ,*

$$B(y'_1 - y'_2, (l^{-1} - m)\rho) \subset (F_1 - F_2)(B(x', \rho)).$$

**Proof.** Apply Theorem 3.2, for the special case where  $Y = Z = W$ ,  $G(y, z) := \{y - z\}$  and take into account the discussion in Remark 4.1.  $\square$

On the basis of Theorem 3.2 and Example 4.5 one has the following result.

**Theorem 4.8** *Let  $X$  be a metric space and  $Y$  be a normed vector space,  $F_1 : X \rightrightarrows Y$  and  $F_2 : X \rightrightarrows Y$  be multifunctions and  $(\bar{x}, \bar{y}_1, \bar{y}_2) \in X \times Y \times Y$  such that  $(\bar{x}, \bar{y}_1) \in \text{Gr } F_1$  and  $(\bar{x}, \bar{y}_2) \in \text{Gr } F_2$  and  $\bar{y}_1 \neq \bar{y}_2$ . Suppose that the following assumptions are satisfied:*

- (i)  $\text{Gr } F_1$  and  $\text{Gr } F_2$  are locally complete around  $(\bar{x}, \bar{y}_1)$  and  $(\bar{x}, \bar{y}_2)$ , respectively;
- (ii)  $F_1$  is metrically regular around  $(\bar{x}, \bar{y}_1)$  with constant  $l > 0$ ;
- (iii)  $F_2$  has the Aubin property  $(\bar{x}, \bar{y}_2)$  with constant  $m > 0$ ;
- (iv)  $lm < 1$ .

*Then there exists  $\varepsilon > 0$  such that, for every  $\rho \in (0, \varepsilon)$ ,*

$$B(\|\bar{y}_1 - \bar{y}_2\|, (l^{-1} - m)\rho) \subset \{\|y_1 - y_2\| \mid y_1 \in F_1(x), y_2 \in F_2(x), x \in B(\bar{x}, \rho)\}.$$

*Moreover, there exists  $\varepsilon' > 0$  such that, for every  $\rho \in (0, \varepsilon')$ , and every  $(x', y'_1, y'_2) \in \text{Gr}(F_1, F_2) \cap [B(\bar{x}, \varepsilon') \times B(\bar{y}_1, \varepsilon') \times B(\bar{y}_2, \varepsilon')]$ ,*

$$B(\|y'_1 - y'_2\|, (l^{-1} - m)\rho) \subset \{\|y_1 - y_2\| \mid y_1 \in F_1(x), y_2 \in F_2(x), x \in B(x', \rho)\}.$$

In the same line, taking into account Example 4.6 instead of Example 4.5, one can formulate a similar result on Riemannian manifolds.

Our intention is to obtain next some fixed point assertions, which are equivalent, on Banach spaces, to the set-valued Lyusternik-Graves Theorem. In [5, Theorem 5], a more general variant



of the next result concerning fixed points, stated on metric spaces, is shown to take place. In the same paper, it is proved that this result can be used to obtain the set-valued Lyusternik-Graves Theorem on metric spaces. The converse is also true on Banach spaces, as proved in [8, Theorem 4.4].

We provide next another proof of this result, when the output space is a normed vector space.

**Theorem 4.9** *Let  $X$  be a metric space,  $Y$  be a normed vector space,  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \rightrightarrows Y$  be multifunctions and  $(\bar{x}, \bar{y}) \in X \times Y$  such that  $(\bar{x}, \bar{y}) \in \text{Gr } F_1 \cap \text{Gr } F_2$ . Suppose that the following assumptions are satisfied:*

- (i)  $\text{Gr } F_1$  and  $\text{Gr } F_2$  are locally complete around  $(\bar{x}, \bar{y})$ ;
- (ii)  $F_1$  is metrically regular around  $(\bar{x}, \bar{y}_1)$  with constant  $l > 0$ ;
- (iii)  $F_2$  has the Aubin property  $(\bar{x}, \bar{y}_2)$  with constant  $m > 0$ ;
- (iv)  $lm < 1$ .

*Then there exist  $\alpha, \beta > 0$  such that, for every  $x \in B(\bar{x}, \alpha)$ , one has*

$$d(x, \text{Fix}(F_1^{-1}F_2)) \leq (l^{-1} - m)^{-1}d(F_1(x) \cap B(\bar{y}, \beta), F_2(x)), \quad (4.2)$$

where

$$\text{Fix}(F_1^{-1}F_2) := \{x \in X \mid F_1(x) \cap F_2(x) \neq \emptyset\}.$$

**Proof.** Our intention is to apply Theorem 4.8 in order to get (4.2). Using the assumptions made, one can find  $\gamma > 0$  such that the assumptions 1.-5. from the beginning of the proof of Theorem 3.2 are satisfied with  $\alpha$  replaced by  $\gamma$ ,  $L^{-1}$  replaced by  $l$ ,  $M$  replaced by  $m$ ,  $C = D = 1$  and  $G$  as in Remark 4.3. Next, choose  $\varepsilon > 0$  as in (3.8), but with  $2^{-1}\gamma$  instead of  $\alpha$ .

Now, take  $\rho \in (0, \min\{\varepsilon, (6^{-1}(l^{-1} - m)^{-1}\gamma)\})$ ,  $\beta := 2^{-1}(l^{-1} - m)\rho$  and  $\alpha > 0$  such that  $\alpha < \rho, m\alpha < \beta$ . Finally, fix arbitrary  $x \in B(\bar{x}, \alpha)$ . Because  $\alpha < \rho < \varepsilon < 2^{-1}\gamma$ , one has that  $B(x, \frac{\gamma}{2}) \subset B(\bar{x}, \gamma)$ . Remark that if  $F_1(x) \cap B(\bar{y}, \beta) = \emptyset$  or  $F_2(x) = \emptyset$  or  $F_1(x) \cap B(\bar{y}, \beta) \cap F_2(x) \neq \emptyset$ , relation (4.2) trivially holds. Suppose next that none of the previous relations is satisfied, and take arbitrary  $y_1 \in F_1(x) \cap B(\bar{y}, \beta)$ . Then  $D(y_1, \frac{\gamma}{2}) \subset D(\bar{y}, \gamma)$ . Also, for every  $\mu > 0$ , there exists  $y_2^\mu \in F_2(x)$  (hence  $y_2^\mu \neq y_1$ ) such that

$$0 < \|y_1 - y_2^\mu\| < d(y_1, F_2(x)) + \mu. \quad (4.3)$$

Using (iii), one has that

$$d(\bar{y}, F_2(x)) \leq e(F_2(\bar{x}) \cap V, F_2(x)) \leq md(x, \bar{x}).$$

Hence, for arbitrary  $\mu > 0$ , there exists  $v_\mu \in F_2(x)$  such that

$$\|v - \bar{y}\| \leq md(x, \bar{x}) + \mu \leq m\alpha + \mu.$$

Then, for  $\mu > 0$  sufficiently small,  $\|v - \bar{y}\| < \beta$ . Now, because  $d(y_1, F_2(x)) \leq \|y_1 - v\| < 2\beta$ , for  $\mu > 0$  sufficiently small, we have also that  $d(y_1, F_2(x)) + \mu < 2\beta$ . Then  $\|y_1 - y_2^\mu\| < 2\beta$  and, furthermore,

$$\|y_2^\mu - \bar{y}\| \leq \|y_2^\mu - y_1\| + \|y_1 - \bar{y}\| \leq 3\beta < 2^{-1}\gamma.$$

Consequently, for  $\mu$  small enough,  $D(y_2^\mu, \frac{\gamma}{2}) \subset D(\bar{y}, \gamma)$ .

Denote  $\rho_0 := (l^{-1} - m)^{-1}[d(y_1, F_2(x)) + \mu] > 0$  and remark that  $\rho_0 < 2(l^{-1} - m)^{-1}\beta = \rho < \varepsilon$  for  $\mu$  sufficiently small. Then all the assumptions of Theorem 4.8 are satisfied for the reference points  $x, y_1, y_2^\mu$ , so one can infer that

$$0 \in B(\|y_1 - y_2^\mu\|, d(y_1, F_2(x)) + \mu) \subset \{\|y'_1 - y'_2\| \mid y'_1 \in F_1(u), y'_2 \in F_2(u), u \in B(x, \rho_0)\}.$$

Hence, there exists  $u \in B(x, \rho_0)$  and  $y'_1 \in F_1(u), y'_2 \in F_2(u)$  such that  $0 = \|y'_1 - y'_2\|$ . Then  $u \in \text{Fix}(F_1^{-1}F_2)$  and

$$d(x, \text{Fix}(F_1^{-1}F_2)) \leq d(x, u) < (l^{-1} - m)^{-1}[d(y_1, F_2(x)) + \mu].$$

Making  $\mu \rightarrow 0$ , one gets (4.2). □

This theorem can be putted into relation with some very recent results in literature (see [5, Theorem 5], [12, Theorems 2, 3], [8, Theorem 4.4] and the references therein).

## 5 Applications

### 5.1 Local stability of compositions

In this section we further investigate some general consequences of Theorem 3.2 on parametric systems. To this end we recall first a stability notion given in [9].

**Definition 5.1** *Let  $F : X \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be multifunctions and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $\bar{y} \in F(\bar{x})$ ,  $\bar{z} \in G(\bar{x})$ . We say that the multifunction  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{y}, \bar{z})$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for every  $x \in B(\bar{x}, \delta)$  and every  $w \in (F+G)(x) \cap B(\bar{y}+\bar{z}, \delta)$ , there exist  $y \in F(x) \cap B(\bar{y}, \varepsilon)$  and  $z \in G(x) \cap B(\bar{z}, \varepsilon)$  such that  $w = y + z$ .*

This definition has a parametric variant as well.

**Definition 5.2** *Let  $F : X \times P \rightrightarrows Y$ ,  $G : X \rightrightarrows Y$  be multifunctions and  $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$  such that  $\bar{y} \in F(\bar{x}, \bar{p})$ ,  $\bar{z} \in G(\bar{x})$ . We say that the multifunction  $(F, G)$  is locally sum-stable around  $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for every  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$  and every  $w \in (F_p + G)(x) \cap B(\bar{y}+\bar{z}, \delta)$ , there exist  $y \in F_p(x) \cap B(\bar{y}, \varepsilon)$  and  $z \in G(x) \cap B(\bar{z}, \varepsilon)$  such that  $w = y + z$ .*

We have now the tools in order to investigate some particular cases of Theorem 3.2. Consider the case where  $W$  is a linear space with a shift-invariant metric. First, take  $G : Y \times Z \rightrightarrows W$  with separate variables, i.e.

$$G(y, z) = R(y) + T(z),$$

where  $R : Y \rightrightarrows W$  and  $T : Z \rightrightarrows W$ . Observe now that  $H$  in Theorem 3.2 has the form

$$H(x) = R \circ F_1(x) + T \circ F_2(x).$$

For this situation we formulate the following result.

**Theorem 5.3** *Let  $X, Y, Z$  be metric spaces,  $W$  be a metric space with linear structure such that the metric is shift-invariant. Take  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \rightrightarrows Z$ ,  $R : Y \rightrightarrows W$ ,  $T : Z \rightrightarrows W$  and  $(\bar{x}, \bar{y}, \bar{z}, \bar{w}_1, \bar{w}_2) \in X \times Y \times Z \times W \times W$  such that  $\bar{y} \in F_1(\bar{x})$ ,  $\bar{w}_1 \in R(\bar{y})$ ,  $\bar{z} \in F_2(\bar{x})$ ,  $\bar{w}_2 \in T(\bar{z})$ . Suppose that:*

(i)  *$\text{Gr } F_1$ ,  $\text{Gr } F_2$  are locally complete around  $(\bar{x}, \bar{y})$ ,  $(\bar{x}, \bar{z})$ , respectively, and  $\text{Gr } G$  is locally closed around  $((\bar{y}, \bar{z}), \bar{w})$ ;*

(ii)  *$F_1$  is open at linear rate  $L > 0$  around  $(\bar{x}, \bar{y})$ ;*

(iii)  *$R$  is open at linear rate  $C > 0$  around  $(\bar{y}, \bar{w}_1)$ ;*

(iv)  *$F_2$  has the Aubin property around  $(\bar{x}, \bar{z})$  with constant  $M > 0$ ;*

(v)  $T$  has the Aubin property around  $(\bar{z}, \bar{w}_2)$  with constant  $D > 0$ ;

(vi)  $(R, T)$  is locally sum-stable around  $(\bar{z}, \bar{y}, \bar{w}_1, \bar{w}_2)$  in the sense of Definition 5.2 (where consider  $P = Y$  and  $R$  is formally taken as  $R(y, z) = R(y)$ , for all  $(y, z) \in Y \times Z$ , i.e. constant with respect to  $z$ ).

(vii)  $LC - MD > 0$ .

Then there exists  $\varepsilon > 0$  such that for every  $\rho \in (0, \varepsilon)$  such that

$$B(\bar{w}_1 + \bar{w}_2, (LC - MD)\rho) \subset (R \circ F_1 + T \circ F_2)(B(\bar{x}, \rho)).$$

Moreover, there exists  $\varepsilon' > 0$  such that for every  $\rho \in (0, \varepsilon')$  and every  $(x, y, z, w_1, w_2) \in B(\bar{x}, \varepsilon') \times B(\bar{y}, \varepsilon') \times B(\bar{z}, \varepsilon') \times B(\bar{w}_1, \varepsilon') \times B(\bar{w}_2, \varepsilon')$  such that  $y \in F_1(x)$ ,  $w_1 \in R(y)$ ,  $z \in F_2(x)$ ,  $w_2 \in T(z)$

$$B(w_1 + w_2, (LC - MD)\rho) \subset (R \circ F_1 + T \circ F_2)(B(x, \rho)).$$

**Proof.** It is enough to prove that conditions (iii), (v) and (vi) ensure the properties of  $G$ . There exist  $\nu > 0$ ,  $U \in \mathcal{V}(\bar{y})$ ,  $W_1 \in \mathcal{V}(\bar{w}_1)$  such that for every  $\theta \in (0, \nu)$  and every  $(y, w_1) \in \text{Gr } R \cap [U \times W_1]$ ,

$$B(w_1, C\theta) \subset R(B(y, \theta)).$$

Take  $\varepsilon > 0$  such that  $B(\bar{w}_1, \varepsilon) \subset W_1$ . Then, from (vi), there exists  $\delta > 0$  such that for every  $(z, y) \in B(\bar{z}, \delta) \times B(\bar{y}, \delta)$  and every  $w \in (R_y + T)(z) \cap B(\bar{w}, \delta)$ , there exist  $w_1 \in R(y) \cap B(\bar{w}_1, \varepsilon)$  and  $z \in T(z) \cap B(\bar{w}_2, \varepsilon)$  such that  $w = w_1 + w_2$ .

Let  $\mu > 0$  such that  $B(\bar{y}, \mu) \subset U$  and consider  $U' = B(\bar{y}, \min\{\delta, \mu\})$ ,  $V = B(\bar{z}, \delta)$ ,  $W' = B(\bar{w}, \delta)$ . Take  $z \in V$  and  $(y, w) \in \text{Gr } G_z \cap [U' \times W']$ . Then  $w \in (R_y + T)(z) \cap B(\bar{w}, \delta)$  and  $(z, y) \in B(\bar{z}, \delta) \times B(\bar{y}, \delta)$ , whence there exist  $w_1 \in R(y) \cap B(\bar{w}_1, \varepsilon)$  and  $w_2 \in T(z) \cap B(\bar{w}_2, \varepsilon)$  such that  $w = w_1 + w_2$ . Consequently,  $(y, w_1) \in \text{Gr } R \cap [U \times W_1]$ , whence

$$B(w_1, C\theta) \subset R(B(y, \theta)),$$

i.e.

$$B(w_1, C\theta) + w_2 \subset R(B(y, \theta)) + w_2.$$

The shift-invariance of the distance in  $W$  ensures

$$B(w, C\theta) \subset R(B(y, \theta)) + T(z) = G_z(B(y, \theta)).$$

Therefore, we infer that  $G$  is open at linear rate with respect to  $y$  uniformly in  $z$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $C > 0$ .

We have to prove now that  $G$  has the Aubin property with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), \bar{w})$  with constant  $D > 0$ . We know that there exists  $V \in \mathcal{V}(\bar{z})$  and  $W_2 \in \mathcal{V}(\bar{w}_2)$  such that for all  $z_1, z_2 \in V$ ,

$$e(T(z_1) \cap W_2, T(z_2)) \leq Dd(z_1, z_2).$$

Take  $\varepsilon > 0$  such that  $B(\bar{w}_2, \varepsilon) \subset W_2$ . Take now  $U = B(\bar{y}, \delta)$ ,  $V' = V \cap B(\bar{z}, \delta)$  and  $W' = B(\bar{w}, \delta)$ , where, as above,  $\delta$  is the positive value given by Definition 5.2 for the prescribed  $\varepsilon > 0$ . Take  $z_1, z_2 \in V'$  and  $w \in (R(y) + T(z_1)) \cap W'$ . Then  $w \in (R_y + T)(z_1) \cap B(\bar{w}, \delta)$  and  $(z_1, y) \in B(\bar{z}, \delta) \times B(\bar{y}, \delta)$ . Hence, there exist  $w_1 \in R(y) \cap B(\bar{w}_1, \varepsilon)$  and  $w_2 \in T(z_1) \cap B(\bar{w}_2, \varepsilon)$  such that  $w = w_1 + w_2$ . We deduce that  $w_2 \in T(z_1) \cap W_2$ , whence  $d(w_2, T(z_2)) \leq Dd(z_1, z_2)$ . Then

$$\begin{aligned} d(w, G_y(z_2)) &= d(w_1 + w_2, R(y) + T(z_2)) \\ &\leq d(w_1 + w_2, w_1 + T(z_2)) = d(w_2, T(z_2)) \leq Dd(z_1, z_2). \end{aligned}$$

Since  $w \in (R_y + T)(z_1) \cap B(\bar{w}, \delta)$  was arbitrarily chosen,

$$e(G_y(z_1) \cap W', G_y(z_2)) \leq Dd(z_1, z_2)$$

and the thesis is proved. The final conclusion is a direct application of Theorem 3.2.  $\square$

In order to treat a similar situation in a different fashion we intend to apply Theorem 4.7 for  $R \circ F_1$  and  $-T \circ F_2$ . Since the metric regularity of  $R \circ F_1$  take place provided that  $R, F_1$  share the same property (see the proof of Theorem 5.8), we look now at the Aubin property of  $T \circ F_2$  and we present a definition.

**Definition 5.4** *Let  $F : X \rightrightarrows Y$ ,  $G : Y \rightrightarrows Z$  be multifunctions and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z$  such that  $\bar{y} \in F(\bar{x})$ ,  $\bar{z} \in G(\bar{y})$ . We say that the pair of multifunctions  $F, G$  is locally composition-stable around  $(\bar{x}, \bar{y}, \bar{z})$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, for every  $x \in B(\bar{x}, \delta)$  and every  $z \in (G \circ F)(x) \cap B(\bar{z}, \delta)$ , there exists  $y \in F(x) \cap B(\bar{y}, \varepsilon)$  such that  $z \in G(y)$ .*

We provide next the main reason for introducing this notion.

**Lemma 5.5** *Let  $F : X \rightrightarrows Y$ ,  $G : Y \rightrightarrows Z$  be multifunctions and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z$  such that  $\bar{y} \in F(\bar{x})$ ,  $\bar{z} \in G(\bar{y})$ . If  $F$  and  $G$  have the Aubin property around  $(\bar{x}, \bar{y})$  and  $(\bar{y}, \bar{z})$ , respectively, and  $F, G$  are locally composition-stable around  $(\bar{x}, \bar{y}, \bar{z})$ , then the multifunction  $G \circ F$  has the Aubin property around  $(\bar{x}, \bar{z})$ , and*

$$\text{lip}(G \circ F)(\bar{x}, \bar{z}) \leq \text{lip} F(\bar{x}, \bar{y}) \cdot \text{lip} G(\bar{y}, \bar{z}).$$

**Proof.** According to the assumptions made on  $F$  and  $G$ , one can find  $\alpha > 0$ ,  $l_F > 0, l_G > 0$  such that, for every  $x, u \in B(\bar{x}, \alpha)$ , and every  $y, v \in B(\bar{y}, \alpha)$ , one has

$$e(F(x) \cap B(\bar{y}, \alpha), F(u)) \leq l_F d(x, u), \quad (5.1)$$

$$e(G(y) \cap B(\bar{z}, \alpha), G(v)) \leq l_G d(y, v). \quad (5.2)$$

Applying the local stability of  $F, G$  for  $\varepsilon := 2^{-1}\alpha$ , one can find  $\delta \in (0, \min\{\alpha, (8l_F)^{-1}\alpha\})$  such that the property from Definition 5.4 is satisfied. Take now arbitrary  $x, u \in B(\bar{x}, \delta)$  and  $z \in (G \circ F)(x) \cap B(\bar{z}, \delta)$ . Then there exists  $y \in F(x) \cap B(\bar{y}, 2^{-1}\alpha)$  such that  $z \in G(y)$ . So, according to (5.1), one has that

$$d(y, F(u)) \leq l_F d(x, u).$$

Hence, for every  $\theta \in (0, 4^{-1}\alpha)$ , there exists  $v \in F(u)$  such that

$$d(y, v) \leq l_F d(x, u) + \theta < \frac{\alpha}{2}.$$

In conclusion,  $y, v \in B(\bar{y}, \alpha)$ , so one can apply (5.2) to get that

$$d(z, (G \circ F)(u)) \leq d(z, G(v)) \leq l_G d(y, v) \leq l_G l_F d(x, u) + \theta l_G.$$

But, because of the arbitrariness of  $z$  from  $(G \circ F)(x) \cap B(\bar{z}, \delta)$ , one gets that for every  $x, u \in B(\bar{x}, \delta)$ ,

$$e((G \circ F)(x) \cap B(\bar{z}, \delta), (G \circ F)(u)) \leq l_G l_F d(x, u) + \theta l_G.$$

Making  $\theta \rightarrow 0$ , one gets the conclusion.  $\square$

In the following, we provide an example of two multifunctions with the Aubin property, but for which their composition fails to satisfy the same property.

**Example 5.6** Take  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  given by

$$F(x) := \begin{cases} [0, 1], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ [0, 1] \cup \{2\}, & \text{if } x = 1 \end{cases}$$

and  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  given by

$$G(y) := \begin{cases} [y, 1), & \text{if } y < 1 \\ \{1\}, & \text{if } y = 1 \\ (1, y], & \text{if } y > 1 \end{cases}.$$

One can easily see that both  $F$  and  $G$  have the Aubin property around  $(\bar{x}, \bar{y}) = (1, 1)$  and  $(\bar{y}, \bar{z}) = (1, 1)$ , respectively. But the multifunction  $G \circ F : \mathbb{R} \rightrightarrows \mathbb{R}$ , given by

$$(G \circ F)(x) = \begin{cases} [0, 1], & \text{if } x \in \mathbb{R} \setminus \{1\} \\ [0, 2], & \text{if } x = 1 \end{cases},$$

does not satisfy the same property around  $(\bar{x}, \bar{z}) = (1, 1)$ . Let us prove this last assertion: suppose by contradiction that there exists  $L > 0$  and  $\alpha \in (0, \min\{1, L^{-1}\})$  such that for any  $x, u \in [1 - \alpha, 1 + \alpha]$

$$(G \circ F)(x) \cap [1 - \alpha, 1 + \alpha] \subset (G \circ F)(u) + L|x - u|[-1, 1]. \quad (5.3)$$

Consider now  $x := 1$  and  $u := 1 + \alpha^2$  such that  $x, u \in [1 - \alpha, 1 + \alpha]$ . Clearly,

$$1 + \alpha \in (G \circ F)(x) \cap [1 - \alpha, 1 + \alpha].$$

Following (5.3), we should have:

$$1 + \alpha \in (G \circ F)(1 + \alpha^2) + L\alpha^2[-1, 1]$$

and, in particular,

$$1 + \alpha \in [-L\alpha^2, 1 + L\alpha^2].$$

But this requires that

$$\alpha \leq L\alpha^2,$$

which contradicts the choice of  $\alpha$ . The contradiction shows that we cannot have the Aubin property of  $G \circ F$ .

Now, taking into account Lemma 5.5, the pair  $F, G$  cannot be locally stable under composition around  $(1, 1, 1)$ . Indeed, pick  $\varepsilon \in (0, 2^{-1})$ . Then for every  $\delta > 0$ , choose  $n \in \mathbb{N}$  such that  $n > \max\{\delta, 1\}$ . Taking now  $x_\delta := 1 \in (1 - \delta, 1 + \delta)$  and  $z_\delta := 1 + n^{-1}\delta \in (G \circ F)(x_\delta) \cap (1 - \delta, 1 + \delta)$ , one can easily see that, for every  $y \in F(x_\delta) \cap (1 - \varepsilon, 1 + \varepsilon) = (1 - \varepsilon, 1]$ ,  $z_\delta \notin G(y)$ .

**Remark 5.7** Let us observe that, if one takes in Definition 5.4  $F : X \rightrightarrows Y \times Y$ ,  $F := (F_1, F_2)$ , where  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \rightrightarrows Y$  are two multifunctions,  $G := g$ , where  $g : Y \times Y \rightarrow Y$  is given by  $g(y, z) := y + z$ , and  $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$  such that  $\bar{y} \in F_1(\bar{x})$ ,  $\bar{z} \in F_2(\bar{x})$ , then the local stability under composition of the pair  $F, G$  around  $(\bar{x}, (\bar{y}, \bar{z}), \bar{y} + \bar{z})$  is just the local sum-stability of  $(F_1, F_2)$  around  $(\bar{x}, \bar{y}, \bar{z})$ . Also, in view of Lemma 5.5, one gets that the sum of two multifunctions with the Aubin property around corresponding points has the Aubin property provided that the two multifunctions are locally sum-stable. For more details in this direction, see [9], Definition 4.2 and the subsequent examples and results.

Putting all these facts together, we are now in position to formulate another theorem for the situation of separate variables.

**Theorem 5.8** *Let  $X, Y, Z$  be metric spaces,  $W$  be a metric space with linear structure such that the metric is shift-invariant. Take  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \rightrightarrows Z$ ,  $R : Y \rightrightarrows W$ ,  $T : Z \rightrightarrows W$  and  $(\bar{x}, \bar{y}, \bar{z}, \bar{w}_1, \bar{w}_2) \in X \times Y \times Z \times W \times W$  such that  $\bar{y} \in F_1(\bar{x})$ ,  $\bar{w}_1 \in R(\bar{y})$ ,  $\bar{z} \in F_2(\bar{x})$ ,  $\bar{w}_2 \in T(\bar{z})$ . Suppose that:*

- (i)  $\text{Gr}(R \circ F_1)$  and  $\text{Gr}(T \circ F_2)$  are locally complete around  $(\bar{x}, \bar{w}_1)$  and  $(\bar{x}, \bar{w}_2)$ , respectively.
- (ii)  $F_1$  is open at linear rate  $L > 0$  around  $(\bar{x}, \bar{y})$ ;
- (iii)  $R$  is open at linear rate  $C > 0$  around  $(\bar{y}, \bar{w}_1)$ ;
- (iv)  $F_2$  has the Aubin property around  $(\bar{x}, \bar{z})$  with constant  $M > 0$ ;
- (v)  $T$  has the Aubin property around  $(\bar{z}, \bar{w}_2)$  with constant  $D > 0$ ;
- (vi)  $F_2, T$  are locally composition-stable at  $(\bar{x}, \bar{z}, \bar{w}_2)$ ;
- (vii)  $LC - MD > 0$ .

*Then there exists  $\varepsilon > 0$  such that for every  $\rho \in (0, \varepsilon)$  such that*

$$B(\bar{w}_1 + \bar{w}_2, (LC - MD)\rho) \subset (R \circ F_1 + T \circ F_2)(B(\bar{x}, \rho)).$$

*Moreover, there exists  $\varepsilon' > 0$  such that for every  $\rho \in (0, \varepsilon')$  and every  $(x, y, z, w_1, w_2) \in B(\bar{x}, \varepsilon') \times B(\bar{y}, \varepsilon') \times B(\bar{z}, \varepsilon') \times B(\bar{w}_1, \varepsilon') \times B(\bar{w}_2, \varepsilon')$  such that  $y \in F_1(x)$ ,  $w_1 \in R(y)$ ,  $z \in F_2(x)$ ,  $w_2 \in T(z)$*

$$B(w_1 + w_2, (LC - MD)\rho) \subset (R \circ F_1 + T \circ F_2)(B(x, \rho)).$$

**Proof.** Remark that, using (ii) and (iii), the multifunction  $R \circ F_1$  is  $LC$ -open around  $(\bar{x}, \bar{w}_1)$ . Also, from (iv), (v) and (vi), using Lemma 5.5, one gets that  $T \circ F_2$  has the Aubin property around  $(\bar{x}, \bar{w}_2)$  with constant  $MD$ . Next, consider  $G$  as in Remark 4.1 and apply Theorem 4.7 for  $R \circ F_1$  and  $-T \circ F_2$ .  $\square$

Remark that the main differences between Theorems 5.3 and 5.8 are, on one side, those referring to the completeness and the closedness of the graphs, and, on the other side, those concerning the local stability.

## 5.2 Parametric variational systems

In the sequel, we shall need a result previously given in [9], which makes the link between a parametric multifunction and the associated solution map, providing also interesting metric evaluations and relations between the regularity moduli of involved set-valued mappings. To this aim, consider a multifunction  $H : X \times P \rightrightarrows W$ , where  $X, P$  are metric spaces, and  $W$  is a normed vector space and define the implicit solution map  $S : P \rightrightarrows X$  by

$$S(p) := \{x \in X \mid 0 \in H(x, p)\}.$$

The next implicit multifunction theorem will play an important role since it will provide both metric regularity and Aubin property for  $S$ . The full version of this result is done in [9, Theorem 3.6].

**Theorem 5.9** *Let  $X, P$  be metric spaces,  $Y$  be a normed vector space,  $H : X \times P \rightrightarrows W$  be a set-valued map and  $(\bar{x}, \bar{p}, 0) \in \text{Gr } H$ .*

(i) If  $H$  is open at linear rate  $c > 0$  with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\alpha, \beta, \gamma > 0$  such that, for every  $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \beta)$ ,

$$d(x, S(p)) \leq c^{-1} d(0, H(x, p) \cap B(0, \gamma)). \quad (5.4)$$

Suppose, in addition to (5.4), that  $H$  has the Aubin property with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ . Then  $S$  has the Aubin property around  $(\bar{p}, \bar{x})$  and

$$\text{lip } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0). \quad (5.5)$$

(ii) If  $H$  is open at linear rate  $c > 0$  with respect to  $p$  uniformly in  $x$  around  $(\bar{x}, \bar{p}, 0)$ , then there exist  $\alpha, \beta, \gamma > 0$  such that, for every  $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \beta)$ ,

$$d(p, S^{-1}(x)) \leq c^{-1} d(0, H(x, p) \cap B(0, \gamma)). \quad (5.6)$$

Suppose, in addition to (5.6), that  $H$  has the Aubin property with respect to  $x$  uniformly in  $p$  around  $(\bar{x}, \bar{p}, 0)$ . Then  $S$  is metrically regular around  $(\bar{p}, \bar{x})$  and

$$\text{reg } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{lip}}_x H((\bar{x}, \bar{p}), 0). \quad (5.7)$$

Take  $H(x, p) := G(F_1(x), F_2(x, p))$ , with  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \times P \rightrightarrows Z$ ,  $G : Y \times Z \rightrightarrows W$ . Then the openness result in Theorem 3.2 and the previous implicit multifunction theorem come into play, to ensure results concerning the well-posedness of the solution mapping associated to the next parametric variational system

$$0 \in G(F_1(x), F_2(x, p)). \quad (5.8)$$

Also, for two multifunctions  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \times P \rightrightarrows Z$ , we consider (as in Theorem 5.3 (vi)) the multifunction  $(F_1, F_2) : X \times P \rightrightarrows Y \times Z$  given by

$$(F_1, F_2)(x, p) := F_1(x) \times F_2(x, p).$$

We are now in position to formulate our results concerning the metric regularity and the Aubin property of the solution mapping associated to (5.8).

Practically, we follow the same way as in [9, Theorems 4.12, 4.13], this time on more general setting and parametric systems. This approach was recently brought into attention by the works of Dontchev and Rockafellar [6] and Aragón Artacho and Mordukhovich [1], [2]. Finally, let us remark that the estimations we obtain here cover those in the quoted papers.

**Theorem 5.10** *Let  $X, P, Y, Z$  be metric spaces,  $W$  be a normed vector space,  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \times P \rightrightarrows Z$ ,  $G : Y \times Z \rightrightarrows W$  be set-valued maps and  $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Z$  such that  $\bar{y} \in F_1(\bar{x})$ ,  $\bar{z} \in F_2(\bar{x}, \bar{p})$  and  $0 \in G(\bar{y}, \bar{z})$ . Suppose that the following assumptions are satisfied:*

- (i)  $(F_1, F_2), G$  are locally composition-stable around  $((\bar{x}, \bar{p}), (\bar{y}, \bar{z}), 0)$ ;
- (ii)  $F_1$  has the Aubin property around  $(\bar{x}, \bar{y})$ ;
- (iii)  $F_2$  has the Aubin property with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), \bar{z})$ ;
- (iv)  $F_2$  is metrically regular with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{z})$ ;
- (v)  $G$  is metrically regular with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), 0)$ ;
- (vi)  $G$  has the Aubin property around  $((\bar{y}, \bar{z}), 0)$ .

Then  $S$  is metrically regular around  $(\bar{p}, \bar{x})$ . Moreover, the next relation holds

$$\text{reg } S(\bar{p}, \bar{x}) \leq \widehat{\text{reg}}_p F_2((\bar{x}, \bar{p}), \bar{y}) \cdot \widehat{\text{reg}}_z G((\bar{y}, \bar{z}), 0) \cdot \max\{\text{lip } F_1(\bar{x}, \bar{z}), \widehat{\text{lip}}_x F_2((\bar{x}, \bar{p}), \bar{y})\} \cdot \text{lip } G((\bar{y}, \bar{z}), 0). \quad (5.9)$$



**Proof.** Consider the multifunction  $H : X \times P \rightrightarrows W$  given by

$$H(x, p) := (G \circ (F_1, F_2))(x, p). \quad (5.10)$$

Using (ii) and (iii), one can easily prove that  $(F_1, F_2)$  has the Aubin property with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), (\bar{y}, \bar{z}))$  with modulus  $K := \max\{\text{lip } F_1(\bar{x}, \bar{z}), \widehat{\text{lip}}_x F_2((\bar{x}, \bar{p}), \bar{y})\}$ . In view of (i), (vi) and Lemma 5.5, we know that  $H$  has the Aubin property with respect to  $x$  uniformly in  $p$  around  $((\bar{x}, \bar{p}), 0)$  and the relation

$$\widehat{\text{lip}}_x H((\bar{x}, \bar{p}), 0) \leq K \cdot \text{lip } G((\bar{y}, \bar{z}), 0)$$

holds.

Using now Theorem 2.2, (iv) is equivalent to the fact that  $F_2$  is open at linear rate with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{z})$  and  $\widehat{\text{lop}}_p F_2((\bar{x}, \bar{p}), \bar{y}) = (\widehat{\text{reg}}_p F_2((\bar{x}, \bar{p}), \bar{y}))^{-1}$ . Similarly,  $G$  is open at linear rate with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), 0)$  and  $\widehat{\text{lop}}_z G((\bar{y}, \bar{z}), 0) = (\widehat{\text{reg}}_z G((\bar{y}, \bar{z}), 0))^{-1}$ .

Consequently, there exist  $\varepsilon, L, C > 0$  such that, for every  $(x, p, y, z, w) \in B(\bar{x}, \varepsilon) \times B(\bar{p}, \varepsilon) \times B(\bar{y}, \varepsilon) \times B(\bar{z}, \varepsilon) \times B(\bar{w}, \varepsilon)$  such that  $(p, z) \in \text{Gr}(F_2)_x$  and  $(z, w) \in \text{Gr } G_y$ , and every  $\rho \in (0, \varepsilon)$ ,

$$B(z, \rho) \subset (F_2)_x(B(p, L^{-1}\rho)), \quad (5.11)$$

$$B(w, C\rho) \subset G_y(B(z, \rho)). \quad (5.12)$$

Using now the local stability from (i), there exists  $\delta \in (0, \varepsilon)$  such that, for every  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$  and every  $w \in (G \circ (F_1, F_2))(x, p) \cap B(0, \delta)$ , there exists  $(y, z) \in [F_1(x) \times F_2(x, p)] \cap [B(\bar{y}, \varepsilon) \times B(\bar{z}, \varepsilon)]$  such that  $w \in G(y, z)$ .

Take now arbitrary  $x \in B(\bar{x}, \delta)$ ,  $(p, w) \in \text{Gr } H_x \cap [B(\bar{p}, \delta) \times B(0, \delta)]$  and  $\rho \in (0, \varepsilon)$ . Then  $w \in H(x, p) \cap B(0, \delta)$ , so there exists  $(y, z)$  as above such that  $(p, z) \in \text{Gr}(F_2)_x \cap [B(\bar{p}, \varepsilon) \times B(\bar{z}, \varepsilon)]$ . Also,  $(z, w) \in \text{Gr } G_y \cap [B(\bar{z}, \varepsilon) \times B(\bar{w}, \varepsilon)]$ . Hence, using (5.11) and (5.12),

$$B(w, C\rho) \subset G(y, B(z, \rho)) \subset G(y, F_2(x, B(p, L^{-1}\rho))) \subset H(x, B(p, L^{-1}\rho)).$$

In conclusion,  $H$  is open at linear rate with respect to  $p$  uniformly in  $x$ , and  $\widehat{\text{lop}}_p H((\bar{x}, \bar{p}), 0) \leq \widehat{\text{lop}}_p F_2((\bar{x}, \bar{p}), \bar{y}) \cdot \widehat{\text{lop}}_z G((\bar{y}, \bar{z}), 0) = (\widehat{\text{reg}}_p F_2((\bar{x}, \bar{p}), \bar{y}))^{-1} \cdot (\widehat{\text{reg}}_z G((\bar{y}, \bar{z}), 0))^{-1}$ .

Now the result follows from Theorem 5.9 (ii).  $\square$

Next, we present a more involved result, which makes use of the Theorem 3.2.

**Theorem 5.11** *Let  $X, P, Y, Z$  be metric spaces,  $W$  be a normed vector space,  $F_1 : X \rightrightarrows Y$ ,  $F_2 : X \times P \rightrightarrows Z$ ,  $G : Y \times Z \rightrightarrows W$  be set-valued maps and  $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Z$  such that  $\bar{y} \in F_1(\bar{x})$ ,  $\bar{z} \in F_2(\bar{x}, \bar{p})$  and  $0 \in G(\bar{y}, \bar{z})$ . Suppose that the following assumptions are satisfied:*

- (i)  $(F_1, F_2), G$  is locally composition-stable around  $((\bar{x}, \bar{p}), (\bar{y}, \bar{z}), 0)$ ;
- (ii)  $\text{Gr } F_1$  is complete,  $\text{Gr}(F_2)_p$  is complete for every  $p$  in a neighborhood of  $\bar{p}$ , and  $\text{Gr } G$  is closed;
- (iii)  $F_1$  is open at linear rate around  $(\bar{x}, \bar{y})$ ;
- (iv)  $F_2$  has the Aubin property around  $((\bar{x}, \bar{p}), \bar{z})$ ;
- (v)  $G$  is open at linear rate with respect to  $y$  uniformly in  $z$  around  $((\bar{y}, \bar{z}), 0)$ ;
- (vi)  $G$  has the Aubin property with respect to  $z$  uniformly in  $y$  around  $((\bar{y}, \bar{z}), 0)$ ;

(vii)  $\widehat{\text{lip}}_x F_2((\bar{x}, \bar{p}), \bar{y}) \cdot \widehat{\text{lip}}_z G((\bar{y}, \bar{z}), 0) < \widehat{\text{lop}}_y G((\bar{y}, \bar{z}), 0) \cdot \text{lop } F_1(\bar{x}, \bar{z})$ .

Then  $S$  has the Aubin property around  $(\bar{p}, \bar{x})$ . Moreover, the next relation is satisfied

$$\text{lip } S(\bar{p}, \bar{x}) \leq \frac{\widehat{\text{lip}}_p F_2((\bar{x}, \bar{p}), \bar{y}) \cdot \widehat{\text{lip}}_z G((\bar{y}, \bar{z}), 0)}{\widehat{\text{lop}}_y G((\bar{y}, \bar{z}), 0) \cdot \text{lop } F_1(\bar{x}, \bar{z}) - \widehat{\text{lip}}_x F_2((\bar{x}, \bar{p}), \bar{y}) \cdot \widehat{\text{lip}}_z G((\bar{y}, \bar{z}), 0)}. \quad (5.13)$$

**Proof.** Take  $L > \text{lop } F_1(\bar{x}, \bar{y})$ ,  $C > \widehat{\text{lop}}_y G((\bar{y}, \bar{z}), 0)$ ,  $M > \widehat{\text{lip}}_x F_2((\bar{x}, \bar{p}), \bar{y})$  and  $D > \widehat{\text{lip}}_z G((\bar{y}, \bar{z}), 0)$  such that  $LC - MD > 0$ .

Now, we intend to prove that there exist  $\tau, t, \gamma > 0$  such that, for every  $(x, p) \in B(\bar{x}, \tau) \times B(\bar{p}, t)$ ,

$$d(x, S(p)) \leq (LC - MD)^{-1} d(0, H(x, p) \cap B(0, \gamma)). \quad (5.14)$$

Using assumptions (ii)-(vi), one can find  $\alpha > 0$  such that:

1.  $\text{Gr } F_1 \cap [D(\bar{x}, \alpha) \times D(\bar{y}, \alpha)]$  is complete; for every  $p \in B(\bar{p}, \alpha)$ ,  $\text{Gr}(F_2)_p \cap [D(\bar{x}, \alpha) \times D(\bar{z}, \alpha)]$  is complete;  $\text{Gr } G \cap [D(\bar{y}, \alpha) \times D(\bar{z}, \alpha) \times D(0, \alpha)]$  is closed.
2. for every  $(x, y) \in B(\bar{x}, \alpha) \times B(\bar{y}, \alpha)$ ,

$$d(x, F_1^{-1}(y)) \leq \frac{1}{L} d(y, F_1(x)) \quad (5.15)$$

3. for every  $p \in B(\bar{p}, \alpha)$  and every  $x, x' \in B(\bar{x}, \alpha)$ ,

$$e(F_2(x, p) \cap B(\bar{z}, \alpha), F_2(x', p)) \leq M d(x, x'). \quad (5.16)$$

4. for every  $(z, w), (z', w') \in B(\bar{z}, \alpha) \times B(0, \alpha)$ ,

$$e(\Gamma(z, w) \cap D(\bar{y}, \alpha), \Gamma(z', w')) \leq \frac{1}{C} (D d(z, z') + d(w, w')). \quad (5.17)$$

5. for every  $y \in B(\bar{y}, \alpha)$  and every  $z, z' \in B(\bar{z}, \alpha)$ ,

$$e(G(y, z) \cap B(0, \alpha), G(y, z')) \leq D d(z, z'). \quad (5.18)$$

Choose now  $\varepsilon > 0$  such that (3.8) are satisfied with  $2^{-1}\alpha$  instead of  $\alpha$ . Finally, apply the property from (i) for  $2^{-1}\alpha$  instead of  $\varepsilon$  and find  $\delta \in (0, 2^{-1}\alpha)$  such that the assertion from Definition 5.4 is true.

Take now  $\rho \in (0, \min\{(LC - MD)^{-1}\delta, \varepsilon\})$ , define  $\gamma := (LC - MD)\rho$  and fix  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$ .

If  $H(x, p) \cap B(0, \gamma) = \emptyset$  or  $0 \in H(x, p) \cap B(0, \gamma)$ , then (5.14) trivially holds. Suppose next that  $0 \notin H(x, p) \cap B(0, \gamma)$ . Then, for every  $\theta > 0$ , one can find  $w_\theta \in H(x, p) \cap B(0, \gamma)$  such that

$$\|w_\theta\| < d(0, H(x, p) \cap B(0, \gamma)) + \theta. \quad (5.19)$$

Because  $d(0, H(x, p) \cap B(0, \gamma)) < (LC - MD)\rho$ , for sufficiently small  $\theta$ ,  $d(0, H(x, p) \cap B(0, \gamma)) + \theta < (LC - MD)\rho$ . Hence, it follows from (5.19) that

$$0 \in B(w_\theta, d(0, H(x, p) \cap B(0, \gamma)) + \theta) \subset B(w_\theta, (LC - MD)\rho) \subset B(w_\theta, \delta), \quad (5.20)$$

so  $w_\theta \in H(x, p) \cap B(0, \delta)$ . Because we have also that  $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$ , one can apply (i) to find  $y_\theta \in F_1(x) \cap B(\bar{y}, 2^{-1}\alpha)$  and  $z_\theta \in F_2(x, p) \cap B(\bar{z}, 2^{-1}\alpha)$  such that  $w_\theta \in G(y_\theta, z_\theta)$ . Consequently,  $B(y_\theta, 2^{-1}\alpha) \subset B(\bar{y}, \alpha)$  and  $B(z_\theta, 2^{-1}\alpha) \subset B(\bar{z}, \alpha)$ .

Observe now that the relations (5.15)-(5.18) are satisfied for  $x, p, y_\theta, z_\theta, w_\theta$  instead of  $\bar{x}, \bar{p}, \bar{y}, \bar{z}, 0$  and  $2^{-1}\alpha$  instead of  $\alpha$ , because every ball centered in these points with radius  $2^{-1}\alpha$  is contained in the initial one with radius  $\alpha$ .

Now, because  $\varepsilon$  was chosen such that (3.8) is satisfied for  $2^{-1}\alpha$  instead of  $\alpha$ , one can use Theorem 3.2 for  $F_1, (F_2)_p, G$ , the reference points  $y_\theta \in F_1(x)$ ,  $z_\theta \in (F_2)_p(x)$ ,  $w_\theta \in G(y_\theta, z_\theta)$ , and for  $\rho_0 := (LC - MD)^{-1} \cdot (d(0, H(x, p) \cap B(0, \gamma)) + \theta) < \rho < \varepsilon$  to get that

$$B(w_\theta, d(0, H(x, p) \cap B(0, \gamma)) + \theta) \subset G \circ (F_1, (F_2)_p)(B(x, \rho_0)).$$

Using also (5.20), we know that  $0 \in G \circ (F_1, (F_2)_p)(B(x, \rho_0))$ , so there exists  $\tilde{x} \in B(x, \rho_0)$  such that  $0 \in G(F_1(\tilde{x}), F_2(\tilde{x}, p))$  or, equivalently,  $\tilde{x} \in S(p)$ . Consequently,

$$d(x, S(p)) \leq d(x, \tilde{x}) < \rho_0 = (LC - MD)^{-1} \cdot (d(0, H(x, p) \cap B(0, \gamma)) + \theta).$$

Making  $\theta \rightarrow 0$ , one gets (5.14).

Now, for the final step of the proof, observe that, from the Aubin property of  $F_2$  with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), \bar{z})$ , one can find  $\beta, k > 0$  such that for every  $x \in B(\bar{x}, \beta)$ , every  $p_1, p_2 \in B(\bar{p}, \beta)$ , one has

$$e(F_2(x, p_1) \cap B(\bar{z}, \beta), F_2(x, p_2)) \leq kd(p_1, p_2). \quad (5.21)$$

Denote  $\xi := \min\{\alpha, \beta\}$ . One can use now (i) for  $2^{-1}\xi$  instead of  $\varepsilon$  to find  $\delta' \in (0, \min\{\xi, \delta, 6^{-1}k^{-1}\xi\})$  such that the assertion from Definition 5.4 is true. Take now arbitrary  $x \in B(\bar{x}, \delta')$ ,  $p_1, p_2 \in B(\bar{p}, \delta')$  and  $w \in H(x, p_1) \cap B(0, \delta')$ . Then there exist  $y \in F_1(x) \cap B(\bar{y}, 2^{-1}\xi)$  and  $z \in F_2(x, p_1) \cap B(\bar{z}, 2^{-1}\xi)$  such that  $w \in G(y, z)$ . Using now (5.21), one obtains that

$$d(z, F_2(x, p_2)) \leq kd(p_1, p_2).$$

Hence, for every  $\mu > 0$ , there exists  $z_\mu \in F_2(x, p_2)$  such that  $d(z, z_\mu) < kd(p_1, p_2) + \mu$ . Then

$$d(z_\mu, \bar{z}) \leq d(z_\mu, z) + d(z, \bar{z}) < kd(p_1, p_2) + \mu + 2^{-1}\xi < 2k\delta' + \mu + 2^{-1}\xi < 6^{-1}5\xi + \mu.$$

But this means, for sufficiently small  $\mu$ , that  $z_\mu \in B(\bar{z}, \xi)$ . Consequently, because of (5.18) and taking into account that  $w \in G(y, z) \cap B(0, \delta')$ , we deduce that

$$d(w, G(y, z_\mu)) \leq Dd(z, z_\mu) < Dkd(p_1, p_2) + D\mu.$$

Finally, because  $y \in F_1(x)$  and  $z_\mu \in F_2(x, p_2)$ , one gets that  $d(w, H(x, p_2)) \leq d(w, G(y, z_\mu))$ , so

$$d(w, H(x, p_2)) < Dkd(p_1, p_2) + D\mu.$$

Making  $\mu \rightarrow 0$  and taking into account the arbitrariness of  $w$  from  $H(x, p_1) \cap B(0, \delta')$ , we obtain that  $H$  has the Aubin property with respect to  $p$  uniformly in  $x$  around  $((\bar{x}, \bar{p}), 0)$ . Moreover,  $\widehat{\text{lip}}_p H((\bar{x}, \bar{p}), 0) \leq \widehat{\text{lip}}_p F_2((\bar{x}, \bar{p}), \bar{y}) \cdot \widehat{\text{lip}}_z G((\bar{y}, \bar{z}), 0)$ .

Now the final conclusion follows from Theorem 5.9 (i).  $\square$

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